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DISSIPATION, LORENTZ METRIC AND INFORMATION: A PHENOMENOLOGICAL CALCULUS OF BILINEAR FORMS

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Abstract—The geometry and physics of three symmetric bilinear forms on the description space, which in our phenomenological calculus underlies the representation of complex systems, are discussed. These three bilinear forms represent dissipation, Lorentz metric and information, and their mathematical connections model a variety of physical and biological concepts, such as space and time, communication, causality, aging, discrimination and measurement. The phenomenological calculus of bilinear forms thus provides a unified theory of thermodynamics, relativity and quantum mechanics.

1. INTRODUCTION

This paper continues our epistemological exploration of the phenomenological calculus. Previous papers in the sequence are [1–6]. The phenomenological calculus is, in essence, an algorithm for the synthesis of mathematical representations of complex and highly interacting systems. The metric structure inherent in the algorithm provides relationship connecting representations.

The mathematical object of the phenomenological calculus is the *description space* D, a subspace of the space $T_1^1(H)$ of type-(1,1) tensors over a real Hilbert space H. Members of D are dyadics of the form $\mathbf{R} = a^i F_i$ (Einstein's summation convention), where $a^i \in H^*$ and $F_i \in H$, and are called *response tensors.* Along the way we shall review enough concepts from [1–6] to make the present paper self-contained. The phenomenological calculus has proven to be extremely versatile in its applicability to various biological, physical and chemical topics, and the reader is referred to [1–6] and also [7] for details.

In the following we study three special bilinear forms on a description space and their interpretations in physical terms. The theory of bilinear forms has a long and rich history, highlighted by the work of Legendre, Gauss, Minkowski and Hasse. We shall investigate the implications of this theory in connection with our own phenomenological calculus.

The principal interpretations of the mathematics of bilinear forms in this paper are conveyed through the theory of relativity. Thus this is an attempt to apply our phenomenological calculus to the information concerning the geometric structure of the world, i.e. the structure of spacetime. In so doing, we pass freely between, indeed identify, physical and geometric concepts, hence agreeing implicitly with J. A. Wheeler's celebrated slogan: "Physics is geometry."

2. DYADICS

Let H be a real Hilbert space of dimension n (finite or infinite) with inner product $\langle\cdot, \cdot\rangle$. Let H^* be its dual space, i.e. the space of all continuous linear functionals on H, with corresponding dual inner product $\langle\cdot, \cdot\rangle^*$. A type-(1, 1) tensor on H is a map from $H^* \times H$ to $\mathbb{R}$ which is bilinear, i.e. linear in each of its two arguments. The linear space of type-(1, 1) tensors on H is denoted $T_1^1(H)$.

A *dyad* aF, where $a \in H^*$ and $F \in H$, is a special type-(1, 1) tensor, the action of which is defined by

$$aF(b, G) = \langle a, b\rangle^*\langle F, G\rangle \tag{1}$$

for $b \in H^*$, $G \in H$. Given dyads aF and bG, their *double inner product* is defined via (1), i.e.

$$aF{:}bG = \langle a, b\rangle^*\langle F, G\rangle. \tag{2}$$

A *dyadic* is a finite sum of dyads, hence

$$\mathbf{R} = a^i F_i. \tag{3}$$

(The Einstein summation convention is used throughout this paper.) Clearly $\mathbf{R} \in T_1^1(H)$, and the definition of the double inner product can be extended to dyadics.

Given a basis $\{e^i\}$ for H^*, every type-(1, 1) tensor $\mathbf{R}$ can be represented in terms of a dyadic

$$\mathbf{R} = e^i F_i \tag{4}$$

for some $F_1, F_2, \ldots \in H$. Thus $T_1^1(H)$ can be considered as the collection of dyadics with $e^i \in H^*$ as "coordinates" and $F_i \in H$ as "components". This in fact is simply an alternative way of saying $T_1^1(H) = H \otimes H^*$. Then the double inner product is defined on all of $T_1^1(H)$—and indeed making $T_1^1(H)$ a Hilbert space[3].

If $\{e_j\}$ is the basis for H dual to $\{e^i\}$, then every $\mathbf{R} \in T_1^1(H)$ can be represented by an $n \times n$ real matrix

$$\mathbf{R} = (R_j^i) = (R(e^i, e_j)). \tag{5}$$

Thus, in particular, $T_1^1(H)$ has dimension n^2, and can therefore be identified as the real vector space $\mathbb{R}^{n^2}$, in which the dyadic (5) is identified with the vector

$$\mathbf{R} = (R_1^1, R_1^2, \ldots, R_1^n, R_2^1, \ldots, R_2^n, \ldots, R_n^1, \ldots, R_n^n). \tag{6}$$

Further, the double inner product : of dyadics in $T_1^1(H)$ is identified with the standard dot product $\cdot$ of vectors in $\mathbb{R}^{n^2}$ under the (5) and (6) correspondence. So $\mathbf{R} \in T_1^1(H)$ admits two alternate descriptions at different hierarchical levels as

$$R \in H \otimes H^* \cong \mathbb{R}^n \otimes \mathbb{R}^n \tag{7}$$

and

$$R \in \mathbb{R}^{n^2}. \tag{8}$$

Henceforth we shall make the (isopmorphism) identification $(T_1^1(H), :) = (\mathbb{R}^{n^2}, \cdot)$ with (4) = (5) = (6).

3. BILINEAR FORMS ON DESCRIPTION SPACE

The *description space* D determined by a set of *constitive parameters* $a^1, a^2, \ldots, a^m \in H^*$ is

$$D = \{\mathbf{R} = a^i F_i : F_1, F_2, \ldots, F_m \in H\} \subset T_1^1(H). \tag{9}$$

Members $\mathbf{R}$ of D are called *response tensors*. So response tensors are simply dyadics having a specified form

$$\mathbf{R} = a^i F_i, \tag{10}$$

and belonging to a specified subspace D of $T_1^1(H)$.

If $\{a^i\}$ spans a subspace of H^* of dimension $k \leq n$, then D had dimension kn [2]. In the following we shall assume for convenience $k = n$, i.e. $D = T_1^1(H)$. For "proper" subspaces $D \subset T_1^1(H)$, the term "restricted to D" can be applied to all the operators and forms and the mathematics will remain valid with trivial modifications.

A *bilinear form* (i.e. a (0, 2)-form) on D is a function f, which assigns to each ordered pair of response tensors $\mathbf{R}$, $\mathbf{S}$ in D a scalar $f(\mathbf{R}, \mathbf{S})$ in $\mathbb{R}$, and which is linear as a function of either of its arguments when the other is fixed. The set of all continuous bilinear forms on D is a linear subspace of the space $\mathbb{R}^{D \times D}$ of all functions from $D \times D$ to $\mathbb{R}$, and is denoted by $T_2^0(D)$ $(= D^* \otimes D^*)$.

Using the $(T_1^1(H), :) = (\mathbb{R}^{n^2}, \cdot)$ isomorphism, we can make use of all the results of bilinear forms on vector spaces. (See [8], Chap. 10, for a review.) In particular, $f \in T_2^0(D)$ can be represented by the $n^2 \times n^2$ matrix

$$[f]_A = (f(\boldsymbol{\alpha}_k, \boldsymbol{\alpha}_l)) \tag{11}$$

where $\mathbf{A} = \{\boldsymbol{\alpha}_k\}$ is an ordered basis for $T_1^1(H) = \mathbb{R}^{n^2}$. For example, $\{\boldsymbol{\alpha}_k\}$ can be the "standard basis" $\{\mathbf{E}_j^i = e^i e_j\}$ [defined in eqn (7) of [2]: $\mathbf{E}_j^i$ is an $n \times n$ matrix with 1 at position (i, j) and 0 elsewhere]. Thus, in fact,

$$T_2^0(D) \subset T_2^0(\mathbb{R}^{n^2}). \tag{12}$$

Furthermore, matrices representing the same bilinear form in different bases are *congruent*. In other words, if P is the transition matrix from the basis $\mathbf{A}$ to the basis $\mathbf{B}$, then

$$[f]_\mathbf{B} = P^T [f]_\mathbf{A} \mathrm{P}. \tag{13}$$

4. THREE SPECIAL BILINEAR FORMS

A bilinear form f on D is *symmetric* if for all response tensors $\mathbf{R}$, $\mathbf{S}$ in D,

$$f(\mathbf{R}, \mathbf{S}) = f(\mathbf{S}, \mathbf{R}). \tag{14}$$

The main object of this paper is the study of the following three symmetric bilinear forms on D.

$$\text{(a)} \quad \delta(\mathbf{R}, \mathbf{S}) = \mathbf{R}:\mathbf{S} = \mathrm{tr}(\mathbf{R}\mathbf{S}^T), \tag{15}$$

$$\text{(b)} \quad \lambda(\mathbf{R}, \mathbf{S}) = \mathbf{R}^T:\mathbf{S} = \mathrm{tr}(\mathbf{R}\mathbf{S}), \tag{16}$$

$$\text{(c)} \quad \iota(\mathbf{R}, \mathbf{S}) = \mathrm{tr}(\mathbf{R})\,\mathrm{tr}(\mathbf{S}), \tag{17}$$

where tr denotes the trace function and T denotes the transpose function [of the matrix representation (5) of the respective response tensors]. Since these functions are invariant with respect to a change of bases, we can safely speak of *the* matrix representation (5), independent of the bases $\{e^i\}$, $\{e_j\}$, and the bilinear forms are well defined. It is easy to verify that δ, λ and ι are indeed symmetric, using the properties of tr and T. Further, these properties also ensure that δ, λ and ι are the only three distinct "binary" combinations of $\mathbf{R}$ and $\mathbf{S}$ under tr and T (e.g. $\mathrm{tr}(\mathbf{R}^T\mathbf{S}) = \delta$, $\mathrm{tr}(\mathbf{R}^T\mathbf{S}^T) = \lambda$, etc.).

Associated with each symmetric bilinear form f is a *quadratic form*, a function q from D to $\mathbb{R}$ defined by

$$q(\mathbf{R}) = f(\mathbf{R}, \mathbf{R}). \tag{18}$$

A symmetric bilinear form $f \in T_2^0(D)$ is completely determined by its associated quadratic form, according to the *polarization identity*

$$f(\mathbf{R}, \mathbf{S}) = \tfrac{1}{4}q(\mathbf{R} + \mathbf{S}) - \tfrac{1}{4}q(\mathbf{R} - \mathbf{S}) = \tfrac{1}{2}[q(\mathbf{R} + \mathbf{S}) - q(\mathbf{R}) - q(\mathbf{S})]. \tag{19}$$

The quadratic forms associated with δ, λ and ι are

$$\text{(a)} \quad \delta(\mathbf{R}) = \mathbf{R}:\mathbf{R} = \|\mathbf{R}\|^2, \tag{20}$$

$$\text{(b)} \quad \lambda(R) = \mathbf{R}^T:\mathbf{R} = \mathrm{tr}(\mathbf{R}^2), \tag{21}$$

$$\text{(c)} \quad \iota(\mathbf{R}) = [\mathrm{tr}(\mathbf{R})]^2. \tag{22}$$

We use the same symbols: δ, λ and ι—it is evident from the number of arguments whether the bilinear or the quadratic form is meant.

It is clear that $\delta(\mathbf{R}, \mathbf{S})$ is just the double inner product on D, while $\delta(\mathbf{R})$ is the squared norm. In irreversible thermodynamics δ is the *dissipation* function [1–6], and in quantum mechanics δ is interpreted as the probability of events[6]. The interpretations of λ and ι will follow.

5. MATRIX REPRESENTATIONS OF BILINEAR FORMS

Let $\{e^i\}$ and $\{e_j\}$ be a set of dual bases for H^* and H, and let response tensors be represented by the vector forms (6) in $\mathbb{R}^{n^2}$. Then an ordered basis for $(\mathbb{R}^{n^2}, \cdot)$ is

$$\mathbf{E} = \{\mathbf{E}_1^1, \mathbf{E}_2^1, \ldots, \mathbf{E}_n^1, \mathbf{E}_1^2, \ldots, \mathbf{E}_n^2, \ldots, \mathbf{E}_1^n, \ldots, \mathbf{E}_n^n\}. \tag{23}$$

The matrix representations of our three special symmetric bilinear forms in the ordered basis $\mathbf{E}$ are the following three $n^2 \times n^2$ matrices:

$$\text{(a)} \quad [\delta]_{\mathbf{E}} = 1_{n^2 \times n^2} \text{ (identity matrix)} \tag{24}$$

$$\text{(b)} \quad [\lambda]_{\mathbf{E}} = \begin{bmatrix} E_1^1 & E_1^2 & \cdots & E_1^n \\ E_2^1 & \cdot & & \\ \vdots & & \cdot & \vdots \\ E_n^1 & \cdot & \cdot & \cdot\ E_n^n \end{bmatrix} \tag{25}$$

(where each E_j^i forms an $n \times n$ submatrix of the $n^2 \times n^2$ matrix)

$$\text{(c)} \qquad [\iota]_{\mathbf{E}} = \begin{bmatrix} E_1^1 & E_2^1 & \cdots & E_n^1 \\ E_1^2 & \cdot & & \\ \vdots & & \cdot & \vdots \\ E_1^n & \cdot & \cdot & E_n^n \end{bmatrix}. \tag{26}$$

Note that $[\lambda]_E^2 = 1_{n^2 \times n^2}$, hence (the matrix representation of) λ is a *duality*[4], and that $[\lambda]_{\mathbf{E}}$ and $[\iota]_{\mathbf{E}}$ are mutual "transposes" if we consider them "$n \times n$ matrices with $n \times n$ matrices as entries."

As a running example, let us consider the case $n = 2$. Then

$$[\delta]_{\mathbf{E}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{27}$$

$$[\lambda]_{\mathbf{E}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{28}$$

$$[\iota]_{\mathbf{E}} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}. \tag{29}$$

The *rank* $r(f)$ of a bilinear form f is the rank of its matrix representation in any ordered basis. It is well defined because congruent matrices (13) have the same rank. In particular, we have

$$\text{(a)} \qquad r(\delta) = n^2, \tag{30}$$

$$\text{(b)} \qquad r(\lambda) = n^2, \tag{31}$$

$$\text{(c)} \qquad r(\iota) = 1. \tag{32}$$

Thus both δ and λ have full rank—i.e. they are *nondegenerate* (= *nonsingular*), and hence for every nonzero $\mathbf{R} \in D$ there exists an $\mathbf{S} \in D$ (and for every nonzero $\mathbf{S} \in D$ there exists an $\mathbf{R} \in D$) such that

$$\delta(\mathbf{R}, \mathbf{S}) \neq 0; \tag{33}$$

and similarly for λ. The bilinear form ι, on the other hand, is highly *singular*.

6. DIAGONALIZATION AND SIGNATURE

Let f be a symmetric bilinear form with rank r on a real vector space. Then there is an ordered basis $\{\beta_k\}$ in which the matrix of f is diagonal, and such that

$$f(\beta_k, \beta_k) = \pm 1, \qquad k = 1, \ldots, r. \tag{34}$$

Further, the number f^+ of basis vectors β_k for which $f(\beta_k, \beta_k) = 1$ (hence also the number f^- of basis vectors β_k for which $f(\beta_k, \beta_k) = -1$) is independent of the choice of basis. The difference

$$s(f) = f^+ - f^- \tag{35}$$

is thus an invariant of f, and is called the *signature* of f. (Note also that $r = f^+ + f^-$.) The above is known as the Jacobi–Sylvester Inertia Theorem[8] (Section 10.2).

For our three special bilinear forms, we have

$$\text{(a)} \quad \delta\colon\ \delta^+ = n^2, \qquad \delta^- = 0, \qquad s(\delta) = n^2, \tag{36}$$

$$\text{(b)} \quad \lambda\colon\ \lambda^+ = \tfrac{1}{2}n(n+1), \qquad \lambda^- = \tfrac{1}{2}n(n-1), \qquad s(\lambda) = n, \tag{37}$$

$$\text{(c)} \quad \iota\colon\ \iota^+ = 1, \qquad \iota^- = 0, \qquad s(\iota) = 1. \tag{38}$$

An *inner product* is a symmetric bilinear form f which satisfies

$$q(\mathbf{R}) = f(\mathbf{R}, \mathbf{R}) > 0 \qquad \text{if } \mathbf{R} \neq \mathbf{0}. \tag{39}$$

A bilinear form satisfying (39) is called *positive definite*. In other words an inner product is a positive definite, symmetric bilinear form. This condition (39) is equivalent to $s(f)$ = dimension of base space (which necessarily implies nondegeneracy, r = full rank). So (36) further verifies that $: = \delta$ is indeed an inner product on D (original proof in [2] and also [3]).

A bilinear form f is *nonnegative semidefinite* if for all $\mathbf{R}$

$$q(\mathbf{R}) = f(\mathbf{R}, \mathbf{R}) \geq 0, \tag{40}$$

which is implied by (39) but not vice versa. Condition (40) is equivalent to

$$s(f) = r(f), \tag{41}$$

whence the form ι is nonnegative semidefinite.

A *metric tensor* is a nondegenerate symmetric bilinear form. Both δ and λ are metric tensors. λ, however, does not satisfy (40); therefore it is not an inner product. In particular, the quadratic form λ can take on negative values.

The $n = 2$ example will prove illuminating. In this case the diagonal forms of the matrices are [i.e. matrices (27)–(29) are congruent to]

$$\delta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{42}$$

$$\lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{43}$$

$$\iota = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{44}$$

7. LORENTZ METRIC

The reader will have recognized that expression (43) is the Lorentz metric on Minkowski space. Note that this Lorentz metric is on the four-dimensional "description spacetime" D, *constructed* from the underlying Euclidean space $\mathbb{R}^2$. Since λ is the Lorentz metric when $n = 2$, we shall borrow the terminology and call λ the *Lorentz metric* for all values n. The signature of the base space H is passed on directly to the Lorentz metric—on an n^2-dimensional spacetime D, the base space $H = \mathbb{R}^n$ has signature n and so does λ (37).

A note on terminology is in order. In four-dimensional spacetime, the metric diag($+1$, $+1$, $+1$, -1) is more commonly referred to as the Minkowski metric. A Minkowski metric, however, always denotes diag($+1, \ldots, +1, -1$) with only *one* -1; hence for n^2-dimensional spacetime, signature $= (n^2 - 1) - 1 = n^2 - 2$. Since

$$\lambda = \text{diag}(+1, \ldots, +1, -1, \ldots, -1), \tag{45}$$

with $\frac{1}{2}n(n + 1)$ $+1$'s and $\frac{1}{2}n(n - 1)$ -1's, has signature $s(\lambda) = n$, it is not quite the "Minkowski metric" for $n \neq 2$. So we choose the name "Lorentz metric" for λ. Note that the Lorentz metric λ allows the interesting prospect of $\frac{1}{2}n(n - 1)$-dimensional time. This points to the possibility that in alternative universes of higher-dimensional spacetime, one may have space of more than three dimensions *and* time of more than one dimensions.

Our own four-dimensional spacetime, according to the phenomenological calculus, is thus not fundamental. The "boosting" from 2 to 2^2 dimensions leads to interesting questions in symmetry and duality, and the metaphysical implications seem very rich. In particular, the $3 + 1$ splitting of space and time dimensions is a natural consequence of the theory. We shall, however, not speculate here.

That $\lambda(\mathbf{R})$ can take positive, negative, or zero values for nonzero $\mathbf{R}$ allows us to classify the response tensors in D into three corresponding types. Again borrowing the language from relativity theory, we call all response tensors (i.e. "world lines") $\mathbf{R}$ for which $\lambda(\mathbf{R})$ is negative *timelike*. Similarly response tensors $\mathbf{R}$ for which $\lambda(\mathbf{R})$ is positive are called *spacelike,* while those for which $\lambda(\mathbf{R}) = 0$ are called *null.*

Since the 2^2-dimensional description space D equipped with the Lorentz metric λ is isomorphic to Minkowski spacetime, the results from the mathematical theory of special relativity are transferable to the metric space (D, λ). In particular, given any timelike $\mathbf{R}$, we can always find an ordered basis $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\tau}\}$ for $T^1_1(H) = \mathbb{R}^{2^2}$, in which $\mathbf{R}$ has only a $\boldsymbol{\tau}$-component. This shows that our "proper time" (discussed in [3] and [6]) and "quantum mechanical time"[6] are indeed valid analogues to time in relativity—they are $\delta = \mathbf{R}:\mathbf{R}$, $\delta_{\hat{H}} = \langle \mathbf{R} \mid \hat{H}\mathbf{R} \rangle$ (where $\hat{H}$ is the Hamiltonian operator), and $\lambda = \mathbf{R}^T:\mathbf{R}$, respectively. This analogy also further illustrates the apparent universality of the phenomenological calculus, which in this case yields a definition of *time*, that time is an invariant along world lines, defined via a bilinear form on response tensors. The various manifestations of time result from different metrics on the same description space.

If two response tensors $\mathbf{R}$ and $\mathbf{S}$ (i.e. "events" in the language of relativity theory) are such that $\mathbf{R} - \mathbf{S}$ is timelike, then the fact that there is a coordinate system for D in which $\mathbf{R} - \mathbf{S}$ has only a "time-coordinate" means that there is some observer who sees the two events happening in turn—i.e. he has enough time to get from one event to the other. But if the "spatial" separation of the points becomes too great, then $\mathbf{R} - \mathbf{S}$ becomes null and then spacelike, and an observer cannot be present at both in succession. Points whose separation is timelike are thus in *communication*, in the sense that there is a possibility of *information transfer* from one to the other. This is not so for points that are spacelike separated, in which case the response tensors $\mathbf{R}$ and $\mathbf{S}$ can have no direct *causal influence* on each other.

8. INFORMATION

The concept of information leads us to our third bilinear form, $\iota(\mathbf{R}, \mathbf{S}) = \operatorname{tr}(\mathbf{R}) \operatorname{tr}(\mathbf{S})$. First, let us digress to the paper [9] (which, in turn, is based on [10]), in which a model of enzyme-substrate recognition, discrimination, and catalysis is formulated. A substrate F is considered as an element of the space $C(X)$ of continuous functions on a real interval X, and an enzyme a is considered as an element of the dual space $NBV(X)$ of normalized functions of bounded variation. Recognition is defined as the evaluation of the Stieltjes integral (a *bilinear* functional)

$$\langle F, a \rangle = \int_X F \, da. \tag{46}$$

A proper ordering of function spaces and their duals (discussed in Section 12 of [3]) leads to a formulation of the enzyme-substrate problem over the base Hilbert space $H = L^2(m)$, where m is the Lesbegue measure on X. In other words, enzymes and substrates can be considered as elements of the self-dual Hilbert space $L^2(m)$, the space of all square-integrable functions on the measure space X. Under such formulation, a system of enzymes and substrates can be represented by the response tensor $\mathbf{R} = a^i F_i \in T^1_1(L^2(m))$, and recognition is defined by

$$I = \langle F_i, a^i \rangle \qquad \text{(sum over } i\text{)}. \tag{47}$$

Further, the quantities $\mathbf{R}$ and I are related mathematically, precisely by the relation

$$I = \operatorname{tr}(\mathbf{R}), \tag{48}$$

where $\mathbf{R}$ is the matrix representation (5). In fact, in the language of tensor theory, I is an invariant of the dyadic $\mathbf{R}$, known as the *scalar* of $\mathbf{R}$. Thus, recognition is the scalar of response.

Let $J = \operatorname{tr}(\mathbf{S})$, then

$$\iota(\mathbf{R}, \mathbf{S}) = \operatorname{tr}(\mathbf{R}) \operatorname{tr}(\mathbf{S}) = I \cdot J \tag{49}$$

and

$$\iota(\mathbf{R}) = \operatorname{tr}(\mathbf{R})^2 = I^2 \geqslant 0. \tag{50}$$

The quadratic form $\iota(\mathbf{R})$, which conveys the magnitude of recognition, is a natural candidate for the definition of *information*. Again we generalize and call ι the information for all response tensors, not just those relating to enzyme-substrate systems. This terminology will be further justified in Section 13.

It is interesting to consider the three measures of "distance" between response tensors $\mathbf{R}$ and $\mathbf{S}$: $\delta(\mathbf{R} - \mathbf{S})$, $\lambda(\mathbf{R} - \mathbf{S})$ and $\iota(\mathbf{R} - \mathbf{S})$. δ is dissipation, and $\delta(\mathbf{R} - \mathbf{S})$ measures the difference in *aging* between systems $\mathbf{R}$ and $\mathbf{S}$ [3]. λ is the Lorentz metric, and $\lambda(\mathbf{R} - \mathbf{S})$ is related to *causality* (Section 7). ι is information, and $\iota(\mathbf{R} - \mathbf{S})$ is *discrimination* (i.e. difference in the magnitude of recognition; [9]). Thus these three important concepts are, as were the different measurements of time, simply the various manifestations of the *metric* derived from different bilinear forms on the same description space D.

9. BILINEAR FORMS AND ENDOMORPHISMS

The space $T_2^0(D)$ of continuous bilinear forms f on the description space D can be made into a Hausdorff topological vector space by the topology of uniform convergence on bounded sets. Equivalently the topology of $T_2^0(D)$ can be defined by means of the norm

$$\| f \| = \sup\{f(R, S): \| R \| \leqslant 1, \| S \| \leqslant 1\}. \tag{51}$$

On the other hand, the space $End(D)$ of continuous linear mappings u: $D \to D$ (i.e. endomorphisms on D) also has a Hausdorff linear topology defined by uniform convergence on bounded sets; the topology of $End(D)$ may be defined by means of the norm

$$\| u \| = \sup\{\| u(\mathbf{R}) \|: \| \mathbf{R} \| \leqslant 1\}. \tag{52}$$

Using (51) and (52), one can show that the spaces $T_2^0(D)$ and $End(D)$ are *isometrically isomorphic* in a canonical way. Further, symmetric bilinear forms correspond to self-adjoint operators. (For detailed proof see a text on functional analysis; e.g. [11].) The canonical isometric isomorphism is defined as follows: Given $f \in T_2^0(D)$, for every $\mathbf{R} \in D$ consider the map $\mathbf{R}^*: \mathbf{S} \mapsto f(\mathbf{R}, \mathbf{S})$. Clearly $\mathbf{R}^* \in D^*$, the space of continuous linear functionals on D. Hence by the Riesz representation theorem there exists $\hat{\mathbf{R}} \in D$ such that $\mathbf{R}^*(\mathbf{S}) = \hat{\mathbf{R}}: \mathbf{S}$. Define $u_f \in End(D)$ by u_f: $\mathbf{R} \mapsto \hat{\mathbf{R}}$; i.e.

$$u_f(\mathbf{R}): \mathbf{S} = f(\mathbf{R}, \mathbf{S}). \tag{53}$$

The mapping $f \mapsto u_f$ from $T_2^0(D)$ to $End(D)$ is the required isometric isomorphism.

The equality

$$\| f \| = \| u_f \| \tag{54}$$

allows us to compute the norms of our three symmetric bilinear forms δ, λ and ι by calculating those of the self-adjoint operators u_δ, u_λ and u_ι. The case for δ is trivial. The corresponding equation to (53) is

$$\mathbf{R}: \mathbf{S} = \delta(\mathbf{R}, \mathbf{S}), \tag{55}$$

whence $u_\delta = 1_D$: $\mathbf{R} \mapsto \mathbf{R}$, and $\| \delta \| = \| u_\delta \| = 1$.

For λ, the corresponding equation to (53) is

$$\mathbf{R}^T: \mathbf{S} = \lambda(\mathbf{R}, \mathbf{S}). \tag{56}$$

Thus u_λ: $\mathbf{R} \mapsto \mathbf{R}^T$ and $\| \lambda \| = \| u_\lambda \| = 1$. Further, $(u_\lambda)^2 = 1_D$: $\mathbf{R} \mapsto \mathbf{R}^T \mapsto (\mathbf{R}^T)^T = \mathbf{R}$; so the self-adjoint operator u_λ corresponding to the Lorentz metric λ is in fact a *duality* (cf. Section 5, where $[\lambda]_{\mathrm{E}}^2 = 1_{n^2 \times n^2}$).

The case for ι is slightly more complicated. We have

$$\iota(\mathbf{R}, \mathbf{S}) = \mathrm{tr}(\mathbf{R})\,\mathrm{tr}(\mathbf{S}) = u_\iota(\mathbf{R}): \mathbf{S}, \tag{57}$$

where the matrix representation of $u_\iota(\mathbf{R})$ is the scalar matrix $\mathrm{tr}(\mathbf{R})\, 1_{n \times n}$, i.e. an $n \times n$ diagonal matrix whose n diagonal entries are each $\mathrm{tr}(\mathbf{R})$. The norm

$$\| u_\iota \| = \sup\{\| \mathrm{tr}(\mathbf{R})\, 1_{n \times n} \|: \| \mathbf{R} \| \leqslant 1\} \tag{58}$$

can be calculated directly, but there is another method. Note that the function tr is in fact a continuous linear operator from the Hilbert space D to the Hilbert space $\mathbb{R}$, and that $\iota(R, S) = \mathrm{tr}(\mathbf{R})\,\mathrm{tr}(\mathbf{S})$ is the scalar product in $\mathbb{R}$. Under these conditions, the norm of ι in $T_2^0(D)$ is the square of that of tr as a bounded operator from D to $\mathbb{R}$ (lemma 4.56 in [11]). That is,

$$\| u_\iota \| = \| \iota \| = \| \mathrm{tr} \|^2 = [\sup\{\mathrm{tr}(\mathbf{R}): \| \mathbf{R} \| \leqslant 1\}]^2. \tag{59}$$

For D with dimension n^2 (i.e. for $n \times n$ matrices $\mathbf{R}$), $\| \mathrm{tr} \| = n^{1/2}$, whence $\| \iota \| = \| u_\iota \| = n = (\dim D)^{1/2}$.

It is interesting to note that both δ and λ have full rank ($=$ dim D) and norm 1. In fact they are linear isometries; i.e. for all $\mathbf{R} \in D$,

$$\| u_\delta(\mathbf{R}) \| = \| u_\lambda(\mathbf{R}) \| = \| \mathbf{R} \|. \tag{60}$$

On the other hand, ι is highly singular (rank $= 1$); but, to "compensate," it is an "expansion" operator with norm $= (\dim D)^{1/2}$. Furthermore, the operator u_ι satisfies the equation

$$u_\iota^2 = n u_\iota. \tag{61}$$

Thus the *normalized* operator

$$\bar{u}_\iota = \frac{1}{n} u_\iota, \tag{62}$$

whence

$$[\bar{u}_\iota(\mathbf{R})] = \frac{1}{n} \mathrm{tr}(\mathbf{R})\, 1_{n \times n}, \tag{63}$$

has norm

$$\| \bar{u}_\iota \| = \frac{1}{n} \| u_\iota \| = 1, \tag{64}$$

and is *idempotent*:

$$\bar{u}_\iota^2 = \bar{u}_\iota; \tag{65}$$

so it is (by definition) a *projector* (= projection operator[5]). We shall return to the physical interpretations of this projector in Section 13.

10. RIEMANNIAN DESCRIPTION SPACE

In [2] and [6] we mentioned the possibility of functional dependence of the constitutive parameters $\{a^i\}$ upon the forces $\{F_i\}$, i.e. $a^i = a^i(F)$. This creates a "Riemannian description space" M based on D as a tensor space (7), just as the transition from Euclidean to Riemannian geometry is accomplished by allowing the local coordinates $\{\mathbf{e}^i\}$, hence the metric tensor g, to depend on the components [i.e. $g^{ij}(\mathbf{x}) = \mathbf{e}^i(\mathbf{x})\cdot\mathbf{e}^j(\mathbf{x})$].

A Riemannian geometry can also be constructed based on D as a vector space (8). In special relativity (Section 7) we had a "flat" description space D for spacetime, in which there was an ordered basis $\{\boldsymbol{\alpha}_k\}$ for which the Lorentz metric λ is constant (45) everywhere. In general relativity, because gravitational forces F vary from place to place, spacetime is represented by M, the set of all possible "events," a smooth manifold over D. Since $End(D)$ is a Banach space and $T_2^0(D) \cong End(D)$ (Section 9), $T_2^0(D)$ is also a Banach space, whence the duals of the tangent bundle $T(M)$ and of $T_2^0(T(M))$ both exist. This allows the definition of a metric on M, thus making M into a Riemannian manifold—a *Riemannian description space.*

In M, every event $\mathbf{R} = a^i(F)F_i$ has a neighbourhood that can be given coordinates in a one-one fashion; the coordinate map α taking $\mathbf{R} \in M$ to its coordinates $\{\boldsymbol{\alpha}_k(\mathbf{R})\}$ in the base space D is continuous, and so is its inverse, and different coordinates are related differentiably. The Lorentz metric λ is taken over into general relativity by supposing that at each point $\mathbf{R}$ of M, there is a specified metric tensor $\lambda(\mathbf{R})$ in the tangent space $TM_{\mathbf{R}}$ of M at $\mathbf{R}$. In accordance with the equivalence principle, there exist coordinates $\{\boldsymbol{\alpha}_k(\mathbf{R})\}$ near $\mathbf{R} = a^i(F)F_i$ such that the components of λ at $\mathbf{R}$ in these coordinates take the special relativity form

$$\lambda(\mathbf{R}) = \mathrm{diag}(+1, \ldots, +1, -1, \ldots, -1). \tag{66}$$

The form of the functional dependence of the Lorentz metric λ on $\mathbf{R}$, and hence on the forces F, is the analogue in the phenomenological calculus of the gravitational (i.e. Einstein's field) equations.

11. MORPHISM OF BILINEAR FORMS

A linear operator u on a vector space V, i.e. $u \in End(V)$, defines a linear operator on $V^* = T_1^0(V)$, namely the adjoint $u^* \in End(T_1^0(V))$:

$$u^*(f) = f \circ u \tag{67}$$

for each $f \in T_1^0(V)$ [4] (Section 2). Hence we have the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\;u\;} & V \\ & {\scriptstyle u^*(f)}\searrow \quad \swarrow{\scriptstyle f} & \\ & \mathbb{R} & \end{array} \tag{68}$$

i.e. for every $x \in V$

$$u^*(f)(x) = f \circ u(x) = f(ux). \tag{69}$$

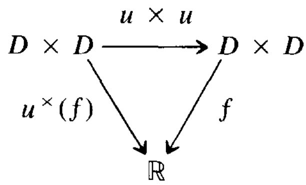
Similarly, given an endomorphism $u \in End(D)$, we can define an endomorphism $u^\times \in End(T_2^0(D))$ on the space of bilinear forms—for each $f \in T_2^0(D)$ let

$$u^\times(f) = f \circ (u \times u), \tag{70}$$

i.e. for every $\mathbf{R}, \mathbf{S} \in D$,

$$u^\times(f)(\mathbf{R}, \mathbf{S}) = f \circ (u \times u)(\mathbf{R}, \mathbf{S}) = f(u\mathbf{R}, u\mathbf{S}). \tag{71}$$

The corresponding commutative diagram is thus



(72)

It is a well-known theorem in linear algebra that in terms of $n^2 \times n^2$ matrix representations $[\cdot]_{\mathbf{E}}$ of the operators.

$$[u^{\times}(f)]_{\mathbf{E}} = [u]_{\mathbf{E}}^{T}[f]_{\mathbf{E}}[u]_{\mathbf{E}}. \tag{73}$$

In particular, when u is nonsingular (i.e. invertible), $u^{\times}(f)$ is congruent to f (cf. (13)), whence

$$[f]_{\mathbf{E}} = [u^{-1}]_{\mathbf{E}}^{T}[u^{\times}(f)]_{\mathbf{E}}[u^{-1}]_{\mathbf{E}} \tag{74}$$

and

$$f = (u^{-1})^{\times}(u^{\times}(f)). \tag{75}$$

12. CATEGORY OF BILINEAR FORMS

For $f, g \in T_2^0(D)$, let

$$\hom(f, g) = \{u \in End(D): g = u^{\times}(f)\}. \tag{76}$$

Note that for specific f and g, $\hom(f, g)$ may be empty, and that when it is nonempty, it may not be a singleton set, because there is the possibility that for $u_1 \neq u_2$.

$$u_1^{\times}(f) = u_2^{\times}(f) = g. \tag{77}$$

The set $\hom(f, f)$ contains all bilinear operators u on D for which

$$f(u\mathbf{R}, u\mathbf{S}) = f(\mathbf{R}, \mathbf{S}) \tag{78}$$

for all $\mathbf{R}$, $\mathbf{S}$ in D. An operator u satisfying (78) is said to *preserve* f, and $u \in \hom(f, f)$ if and only if

$$[f]_{\mathbf{E}} = [u]_{\mathbf{E}}^{T}[f]_{\mathbf{E}}[u]_{\mathbf{E}} \tag{79}$$

[cf. (73)]. If f is a symmetric bilinear form, then u preserves f if an only if u preserves the corresponding quadratic form q [defined by (18)], in the sense that for each $\mathbf{R} \in D$,

$$q(u\mathbf{R}) = q(\mathbf{R}). \tag{80}$$

The identity operator 1_D clearly preserves every bilinear form. The collection of linear operators which preserve a given bilinear form is closed under the formation of (operator) products. Thus $\hom(f, f)$ is in fact a *monoid*. If f is nondegenerate, then every operator

u in hom(f, f) is invertible and u^{-1} is also in hom(f, f). Hence hom(f, f) is a *group* under the operation of composition[8] (Section 10.4).

If $u \in$ hom(f, g) and $v \in$ hom(g, h), then clearly $v \circ u \in$ hom(f, h). Moreover, $u \in$ hom(f, g) is an *isomorphism* (i.e. invertible morphism) if and only if $u \in End(D)$ is invertible [cf. (75)].

The above leads to the inevitable conclusion that $T_2^0(D)$ is a *category*, the objects of which are bilinear forms $f \in T_2^0(D)$ and the morphisms of which are linear operators u in the hom-sets hom(f, g).

13. PHENOMENOLOGICAL CONNECTIONS

In this final section we concentrate on the hom-sets hom(f, g) for f, g = δ, λ, ι, and thereby on the connections among our three special bilinear forms.

Since $[\delta]_{\mathbf{E}} = 1_{n^2 \times n^2}$ (24), the group hom(δ, δ) preserving the dissipation function δ corresponds to the matrices M for which

$$1 = M^T M, \tag{81}$$

i.e. *orthogonal matrices*. Thus the group hom(δ, δ) is in fact the n^2-dimensional *orthogonal (Euclidean) group*.

More generally, the group of matrices hom(f, f) preserving a nondegenerate, symmetric bilinear form f on D is called a *pseudo-orthogonal group*. [When the signature $s(f) = n^2$ we obtain the orthogonal group as a particular type of pseudo-orthogonal group.] Hence the group hom(λ, λ) preserving the Lorentz metric λ is a pseudo-orthogonal group. Since for $n = 2$, λ is the four-dimensional Minkowski metric (43), we shall again borrow the terminology and call, for all values of n, hom(λ, λ) the (n^2-dimensional) *Lorentz group* and morphisms in hom(λ, λ) *Lorentz transformations*.

As the bilinear form ι is degenerate, hom(ι, ι) contains some singular operators and is therefore only a monoid. Because of the information-theoretic connections of ι (Section 8), we shall call hom(ι, ι) the *Shannon monoid*, the structure of which we shall explore in another paper.

The hom-set hom(δ, g) corresponds to the matrices M for which

$$[g]_{\mathbf{E}} = M^T M. \tag{82}$$

But for a self-adjoint operator g, (82) is satisfied for some matrix M if and only if g is nonnegative semidefinite[8] (Section 9.3). Thus we conclude that hom(δ, λ) is an empty set, while hom(δ, ι) is nonempty. For example, when ι is in its diagonal form, the operator $\hat{E}_1^1$—projection onto the first coordinate—is in hom(δ, ι).

Similar algebraic arguments show that the hom-sets hom(λ, δ), hom(ι, δ) and hom(ι, λ) are all empty, while hom(λ, ι) is nonempty [again, $\hat{E}_1^1$, hom(λ, ι)].

That both hom(δ, λ) and hom(λ, δ) are empty, i.e. that there are no transformations between δ and λ, is not unexpected—the positive definite metric δ induces a Euclidean geometry on D, while the indefinite metric λ induces a non-Euclidean geometry (of space-time) on D. The two geometries are not transformable to each other.

That hom(δ, ι) and hom(λ, ι) are nonempty but hom(ι, δ) and hom(ι, λ) are empty requires a bit more explanation. Why is it possible to map the description space D with either Euclidean or non-Euclidean geometry onto information, but not vice versa? A hint is given by the fact that the normalized operator $\bar{u}_\iota$ (62) corresponding to information ι is a projector—i.e. the extraction of information, or the measurement process, is a projection. This is in accordance with the axioms of quantum mehanics ([12], Chap. 3, fifth postulate):

If the measurement of the physical quantity **A** on the system in the state $|\psi\rangle$ gives the result c_α, the state of the system immediately after the measurement is the normalized projection,

$$|\bar{\psi}_\alpha\rangle = \frac{\hat{P}_\alpha|\psi\rangle}{\langle\psi|\hat{P}_\alpha|\psi\rangle^{1/2}}, \tag{83}$$

of $|\psi\rangle$ onto the eigensubspace associated with c_α.

Thus when the first measurement is performed, we extract only part of the information content of the original state $|\psi\rangle$. If we perform a second measurement of **A** immediately after the first one, we always find the same result c_α, since the state of the system immediately before the second measurement is $|\bar{\psi}_\alpha\rangle$, and no longer $|\psi\rangle$: $(|\psi\rangle - |\bar{\psi}_\alpha\rangle)$ has been "lost" in the first measurement. This is the concept of *wave packet reduction* in quantum mechanics.

The above is also consistent with the Shannon formulation of information and entropy, in which *entropy* **H** is defined as the expected value of a collection of individual pieces of *information* $\log_2(1/p_\alpha)$, each with a probability of occurance p_α:

$$\mathbf{H} = \sum_i p_i \log_2\left(\frac{1}{p_i}\right). \tag{84}$$

The analogy is established if we identify δ with **H**, and ι with $\log_2(1/p_\alpha)$. Then clearly dissipation δ projects onto information ι, but δ cannot be reconstructed from a single ι.

We see that the process of measurement, or extraction of information, is a highly singular one. A morphism from δ or λ onto ι reduces the rank (i.e. information content), and is hence not invertible. So there are no morphisms from ι onto δ or λ. For a detailed discussion on projections as representations of phenomena, the reader is referred to [5]. For further quantum-mechanical connections of the phenomenological calculus, the reader is referred to [6].

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