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Research Report

ANY MATERIAL REALIZATION OF THE (M,R)-SYSTEMS MUST HAVE NONCOMPUTABLE MODELS

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Robert Rosen’s (M,R)-systems are a class of relational models with a structure that defines a necessary distinguishing feature of organisms. That feature is an impredicative hierarchy of constraint on the properties of the model that correspond to the closure of an organism’s entailment relations with respect to efficient cause. As a consequence, a computable model cannot be an (M,R)-system. This has been mathematically proven, and hence indisputable. Nevertheless, “computable” implementations of the mappings in an (M,R)-system have been reported. This paper explains the logical impossibility of the existence of these “counterexamples.” In particular, it examines the errors in the construction of one of the most interesting among them. The relevance of this result to neuroscientists is that the same structure of closure to efficient cause is observed in brain dynamics.

Keywords: (M,R)-systems; noncomputability; proof.

1. Introduction

The brain is not a computer. This is a conclusion that may be drawn from Robert Rosen’s work on relational biology. That is the essence of why this paper, dealing with seemingly arcane topics in abstract algebra, is relevant to the readers of a journal dedicated to integrative neuroscience. The connection is made more specific in the Editorial of this issue.

Robert Rosen’s research monograph *Life Itself* [12] directly addresses the basic question of biology: “What is life?”. However, it reaches far beyond the domains of biology and penetrates to the foundations of science itself. Therefore, it is not unanticipated that Rosen’s characterization of life, that a living system is closed to efficient causation, also finds inherited applicability in the brain sciences.

Kerckhoff [6] postulated that “the answer to the question of why do brains behave as they do is that they are entailed by an endogenous structure of causation.” Furthermore, in Ref. 5, he gave a concrete example of a closed-loop hierarchy of causation in brain dynamics. Rosen has also written on the topic of relational biology

of the brain. Indeed, Part II (Chaps. 4–8) of Rosen’s *Essays on Life Itself* [13] is entitled “On Biology and the Mind.” He introduced this part thus:

“The mind-brain problem is somewhat apart from my direct line of inquiry, but it is an important collateral illustration of the circle of ideas I have developed to deal with the life-organism problem. I do not deny the importance of the mind-brain problem; it was simply less interesting to me personally, if for no other reason than that one has to be alive before one is sentient. Life comes before mind, and anything I could say about mind and brain would be a corollary of what I had to say about life. That is indeed the way it has turned out.”

The central theorem in *Life Itself* is that if a closed path of efficient causation exists for a natural system, then it has a model that is not computable. An immediate corollary is that a computable model cannot be an (M,R)-system (metabolism-repair system). (I shall briefly explain (M,R)-systems in Sec. 4.) Both the theorem and corollary were discussed in detail, and proven with mathematical rigor in Ref. 12.

No counterexamples can logically exist for a mathematically proven fact. Yet there have been several published reports of algorithms, claiming to implement the algebraic mappings that characterize an (M,R)-system. One notable instance is found in Goertzel’s essay *Goertzel versus Rosen: Contrasting views on the autopoietic nature of life and mind* [2]. Goertzel claims that one could construct computable systems by using division algebras to fulfil Rosen’s mathematical description of a cell (i.e., an (M,R)-system). In this paper, I shall point out the logical fallacy of the attempted search of such counterexamples, and illustrate the mathematical errors in [2]. I shall show that Goertzel’s constructions place his mappings at the wrong levels in the categorical hierarchy, and therefore what he constructed were, in fact, *not* (M,R)-systems.

2. A Little Mathematical Logic

One of the axiom schemes in the predicate calculus is *universal instantiation*:

$$[\forall x P(x)] \Rightarrow P(c) \tag{2.1}$$

It is simply the proposition that if some property is true for *everything*, then that property is true for any particular thing. The dual axiom scheme is called *existential generalization*:

$$P(c) \Rightarrow [\exists x P(x)] \tag{2.2}$$

It is the trivial statement that if a property is true for one particular thing, then there is *something* for which the property is true.

In mathematics, one can make absolute statements, and in particular, absolute statements about the truth of statements validated by proofs. A correct proof is a rigorous step-by-step justification, presented in such a way that cannot be disputed if one follows the rules of logic and accepts a set of axioms as the basis for the

logic system. To refute a proof, one must show a logical fallacy in the argument in one of these steps. A mathematical proof is absolute; it is categorically more than a judicial “proof” of “beyond a reasonable doubt.” When a general statement $\forall xP(x)$ is proven as the conclusion of a hypothesis, it is usually called a *theorem*. A statement that follows directly from it, such as a specific case $P(c)$, is usually called a *corollary*. When the proof of the theorem $\forall xP(x)$ is verifiably correct, it would be a complete waste of time trying to find counterexamples to the corollary, since logic precludes their existence.

However, in the history of mathematics, the logical impossibility of the existence of specific objects has not deterred efforts to search for them. A famous example traces back to Greek antiquity, that of *trisecting an angle*. The problem of dividing an arbitrary angle into three equal parts, by an unmarked straight edge and a pair of compasses, dates from the fifth century BC in Greece. The solution in the negative, that there exist angles that cannot be trisected, was finally offered by Carl Friedrich Gauss and Pierre Wantzel in the nineteenth century AD [1, 15].

Note that the result says that the constructions are *impossible*, in the sense that none can exist rather than there are no known constructions. It is much simpler to prove that something is possible; one only has to demonstrate a successful method. To prove that something is impossible requires stronger reasoning and the current inability to do so is not a proof. Another reason that it took so long for the problem to be solved is that although the statement arises from geometry, the proof comes from abstract algebra — a subject not born until the nineteenth century.

In the following, I give a few definitions and state some theorems without proofs. Readers interested in pursuing the mathematical details are encouraged to explore this subject — Galois theory. Two good references are Refs. 3 and 4.

A complex number a is *algebraic over the field F* , if there is some (nonzero) polynomial over F of which a is a root. It follows that there exists such a polynomial of smallest degree, called a *minimal polynomial* for a over F . Its degree is called the *degree of a over F* , denoted $\deg_F a$. If F is not specified, then it is understood to be the rational numbers Q . Thus, a number a is *algebraic* if there is some (nonzero) polynomial over Q , of which a is a root, and its degree is $\deg_Q a$.

A real number a is called *constructible* if, starting from a line segment of length 1, one can construct a line segment of length $|a|$ in a finite number of steps using straight edge and compasses. Two important theorems are:

Theorem 2.1. *All constructible numbers are algebraic.*

Theorem 2.2. *If a is constructible, then $\deg_Q a$ must be a power of 2.*

Now let $\forall xP(x)$ of our example be the contrapositive statement of Theorem 2.2:

Theorem 2.3. *If $\deg_Q x$ is not a power of 2, then x is not constructible.*

Theorem 2.3 is true for every algebraic number x , and therefore true for any particular algebraic number c . Given a particular number c , “ $P(c)$ ” is the

statement:

Corollary 2.1. *If $\deg_Q c$ is not a power of 2, then c is not constructible.*

The corollary follows from Theorem 2.3 by universal instantiation.

Returning to the angle trisection problem, there are many angles that can be easily trisected. Note, however, that the problem is about the trisection of an *arbitrary* angle. To show that it is not possible to trisect an arbitrary angle, one only needs to show *one* angle that cannot be trisected. The traditional example is 60° , and the logical steps in the argument are as follows. Since 60° itself (the interior angle of an equilateral triangle) can be easily constructed, if it can be trisected, then a 20° angle can be constructed. If 20° can be constructed, then so can the number $c = \cos 20^\circ$ (since it is the perpendicular projection of 1 from one side of a 20° angle to the other). However, $\deg_Q \cos 20^\circ = 3$; therefore, c is not constructible since 3 is not a power of 2.

This means that whatever clever angle trisection scheme one dreams up, it is bound to fail. The trisection of 60° is a contradiction to “ $P(\cos 20^\circ)$.” It would not be necessary to actually examine the scheme in detail to arrive at this conclusion. However, it may not be *trivial to verify* that the proposed trisection scheme cannot possibly trisect 60° . If the scheme is reasonably clever or sufficiently obscure, then it may very well be difficult to show why it would not work. Nevertheless, we know from the outset that the scheme is erroneous, because the impossibility of trisection has been *proven*, yet this has not deterred generations of amateur geometers. Two hundred years after Gauss, mathematics departments at universities world wide still regularly receive unsolicited submissions of angle trisection schemes from enthusiasts. This is partly because this classical geometric problem is easily stated and understood, whereas its abstract algebraic solution is less readily accessible.

The angle trisection example demonstrates how it is possible in mathematics to prove the non-existence of an entity, or the impossibility of doing something. The method of proof is usually *reductio ad absurdum* or “proof by contradiction”, as in this example. One assumes the opposite of what one wants to prove; in this case, the possibility of doing something, and then shows that contradictory propositions follow from the assumption. This example also illustrates universal instantiation that it is not possible to have a counterexample to $P(c)$, without calling $\forall x P(x)$ into question.

It is worthwhile examining the logical consequences of saying that one has a counterexample to a corollary $P(c)$ of a general theorem $\forall x P(x)$. If the counterexample is correct, it implies the truth of the proposition $\neg P(c)$, where \neg is the “negation” operator, which in turn implies $\exists x \neg P(x)$ by existential generalization, a direct contradiction to $\forall x P(x)$. When a counterexample $\neg P(c)$ is reported, there are only three possibilities:

- (1) The theorem is wrong; i.e., the original proof of the theorem $\forall x P(x)$ is incorrect.
- (2) The proposition that is claimed as $P(c)$ is actually not; i.e., the supposed corollary in fact does not follow from the theorem.

- (3) The statement that is reported as $\neg P(c)$ is actually not; i.e., a counterexample has not been produced.

This mathematical logic must be kept in mind when one examines Goertzel's "computable model" of Rosen's (M,R)-system.

3. Rosen's Theorems

In *Life Itself* [12], Rosen defined the term *simulable* and several of its synonyms. A mapping is simulable if it is "definable by an algorithm." It is variously called *computable*, *effective*, and "*evaluable by a mathematical (Turing) machine*." In Chap. 8 of Ref. 12, he gave the following:

Definition 3.1. A natural system N is a *mechanism* if and only if all of its models are simulable.

(Elsewhere he uses this as the definition of a *simple system* [13].) He then *proved* the following five propositions for a mechanism N :

Theorem 3.1. N has a unique largest model M^{\max} .

Theorem 3.2. N has a (necessarily finite) set $\{M_i^{\min}\}$ of minimal models.

Theorem 3.3. The maximal model is equivalent to the direct sum of the minimal ones,

$$M^{\max} = \sum_i M_i^{\min}$$

and is therefore a synthetic model.

Theorem 3.4. Analytic and synthetic models coincide in the category $\mathcal{C}(N)$ of all models of N ; direct sum = direct product.

Theorem 3.5. Every property of N is fractionable.

Immediately following this, in Chap. 9 of Ref. 12, using these just-proven Theorems 3.1–3.5, Rosen presented a detailed *reductio ad absurdum* argument that proves that certain modes of entailment are not available in a mechanism:

Theorem 3.6. There can be no closed path of efficient causation in a mechanism.

These six Theorems 3.1–3.6 above were *proven* in the rigorous mathematical sense of the word. No logical fallacy in Rosen's proofs has ever been demonstrated. The contrapositive statement of Theorem 3.6 is

Theorem 3.7. If a closed path of efficient causation exists in a natural system N , then N cannot be a mechanism.

Taking Definition 3.1 of mechanism into account, this is equivalent to

Theorem 3.8(a). *If a closed path of efficient causation exists for a natural system N , then it has a model that is not simulable.*

An iteration of “efficient cause of efficient cause” is inherently hierarchical. A closed path of efficient causation must form a hierarchical cycle. Both the hierarchy and the cycle (closed loop) are essential attributes of this closure. In formal systems, hierarchical cycles are manifested by impredicativities, or the inability to internalize every self-referent. The nonsimulable model in Theorem 3.8 (a) contains a hierarchical closed loop that corresponds to the closed path of efficient causation in the natural system being modeled. In other words, it is a formal system with an impredicative loop of inferential entailment. Thus, we also have:

Theorem 3.8(b). *If an impredicative loop of inferential entailment exists for a formal system, then it is not simulable.*

I shall take Theorem 3.8, where part (a) is in the natural domain and part (b) is in the formal domain, as the general “ $\forall xP(x)$ ” statement of relational biology.

A natural system that has a nonsimulable model is defined by Rosen as a *complex system* (Chap. 19 of Ref. 13). A necessary condition for a natural system to be an *organism* is that it is closed to efficient causation (Chap. 1 of Ref. 13). Theorem 3.8 then says an organism must be complex. Complexity, while it is the habitat of life, is not life itself. Rosen wrote (Chap. 1 of Ref. 13):

“To be sure, what I have been describing are necessary conditions, not sufficient ones, for a material system to be an organism. That is, they really pertain to what is not an organism, to what life is not. Sufficient conditions are harder; indeed, perhaps there are none. If so, biology itself is more comprehensive than we presently know.”

Any natural system has many different models that do not contain impredicative structures of inferential entailment. The limitation for a mechanism is that this impoverishment applies to *all* of its models. A mechanism may have a finite sequential hierarchy of efficient causes that does not form a loop, or it may have a “flat” loop of efficient causes that are on the same “hierarchical level.” A necessary property of an organism is that there exists at least *one* model that does contain an impredicative structure of entailment, a hierarchical closed loop that corresponds to the closed path of efficient causation in the organism being modeled.

Rosen, in Chap. 10 of Ref. 12, described in detail a relational model with this impredicative structure that corresponds to the property of “closed to efficient causation” possessed by an organism. This relational model is the (M,R)-system. Rosen demonstrated clearly and rigorously that each map in this relational model arises as a logical conclusion entailed by the existence of all the other maps in the model. (Further details are in Sec. 4.) This corresponds to the fact that in an organism,

each efficient cause is an entailed effect of all the other efficient causes. In Chap. 17 of Ref. 13, Rosen defined a cell, in the sense of an autonomous life form, thus

Definition 3.2. A *cell* is (at least) a material structure that realizes an (M,R)-system.

This means that “having an (M,R)-system as a model” is a necessary condition for a natural system to be an autonomous life form.

For our present purpose, since an (M,R)-system has an impredicative loop of inferential entailment that corresponds to the closed path of efficient causation in an organism, it is a particular case of our general Theorem 3.8. Universal instantiation predicates that it is not simulable. This is our “ $P(c)$ ”:

Corollary 3.1. *Any material realization of the (M,R)-systems must have noncomputable models.*

Incidentally, the statement of Corollary 3.1 is the title of this paper, and is a direct quote from Chap. 17 of Ref. 13. The contrapositive statement is:

Corollary 3.2. *A computable model cannot be an (M,R)-system.*

Many years before deriving the noncomputability of (M,R)-systems as a corollary of the more general theorems in Ref. 12, Rosen proved it directly. It was mentioned in Ref. 11, but the actual proofs appeared in a series of papers on abstract biological systems as sequential machines [7–9]. Kerckhoff pointed out in Ref. 6 that “attempting to implement a hierarchical closed loop of inferential entailment is forbidden in systems programming [14]. This loop structure creates ambiguities and each subprocess in the loop pauses, waiting for another subprocess in the loop to provide disambiguating data. The result is deadlock.” This is a practical verification of the validity of Rosen’s Theorem 3.8 from the field of computer science.

Both the theorem, $\forall xP(x)$, and corollary, $P(c)$, are proven in Ref. 12 with mathematical rigor. Yet, purported counterexamples, $\neg P(c)$, appear in the literature claiming to be computable implementations of (M,R)-systems. Remarkably, none of these developments actually challenges the truth of the more general but less accessible Theorem 3.8, our $\forall xP(x)$, itself. At the end of the previous section, I mentioned that a counterexample $\neg P(c)$ can only lead to three conclusions. Valid mathematical proofs are absolute, and Rosen’s critics produce no claim, much less evidence, in support of conclusions (1) and (2). In Secs. 5 and 6, conclusion (3) is validated : the counterexample is wrong.

4. (M,R)-Systems

Before proceeding further, a brief description of Rosen’s (M,R)-system is needed more for the notation than for the details. The two most important references on (M,R)-systems are Refs. 10 and 11. One should refer to these two papers for the fine nuances.

The simplest (M,R)-system may be represented by the diagram

$$A \xrightarrow{f} B \xrightarrow{\Phi} H(A, B) \quad (4.1)$$

Note the use of adjective, *simplest*, here. Form (4.1) is what Rosen used in almost all of his discussions on (M,R)-systems. A general (M,R)-system is actually a network representing the relationship among metabolism and repair components. It is true that form (4.1) captures the essence of all (M,R)-systems; it is possible to project every abstract (M,R)-system onto this simple form. Nevertheless, one must not lose sight of the network aspect of (M,R)-systems.

There are three ways to show the function or *morphism* (in categorical language), f , with its domain A and codomain B :

$$A \xrightarrow{f} B \quad (4.2)$$

$$f : A \rightarrow B \quad (4.3)$$

$$f \in H(A, B) \quad (4.4)$$

I use them interchangeably.

In diagram (4.1) the function f represents metabolism (enzyme):

$$\boxed{\text{Metabolism } f : A \rightarrow B \quad f \in H(A, B)} \quad (4.5)$$

The function Φ represents the repair (gene). It represents the repair of the metabolism function in the sense that its codomain is $H(A, B)$, the hom set of which f is a member. Essentially, Φ makes new copies of f .

$$\boxed{\text{Repair } \Phi : B \rightarrow H(A, B) \quad \Phi \in H(B, H(A, B))} \quad (4.6)$$

What if the repair components themselves need repairing? New functions representing replication (i.e., that which serve to replicate the repair or genetic components) may be defined. A replication map must have as its codomain the hom set $H(B, H(A, B))$, to which repair functions belong. Thus it must be of the form

$$\beta : Y \rightarrow H(B, H(A, B)) \quad (4.7)$$

for some set Y .

One possible choice, in keeping with the pattern of the emerging hierarchy of maps in diagrams (4.5) and (4.6), is $Y = H(A, B)$; thus

$$\beta : H(A, B) \rightarrow H(B, H(A, B)) \quad (4.8)$$

$$\boxed{\text{Replication } \beta : H(A, B) \rightarrow H(B, H(A, B)) \quad \beta \in H(H(A, B), H(B, H(A, B)))} \quad (4.9)$$

The morphism (4.8) may be combined with the second morphism in (4.1) to give a new (M,R)-system from the old one:

$$B \xrightarrow{\Phi} H(A, B) \xrightarrow{\beta} H(B, H(A, B)) \quad (4.10)$$

This has the property that the “metabolic” part of the system (4.10) is the “repair” part of system (4.1), and the “repair” part of system (4.10) is the “replication” part

of system (4.1) [i.e., form (4.8)]. Indeed, one may sequentially extend this structure *ad infinitum*, the next system being

$$H(A, B) \xrightarrow{\beta} H(B, H(A, B)) \xrightarrow{\Xi} H(H(A, B), H(B, H(A, B))) \quad (4.11)$$

The arrow diagrams may be extended indefinitely in either direction.

If there were no more than this to (M,R)-systems, developing this representation would be pointless. The iteration of “repair the repair” in a hierarchic infinite sequence would be an absurd model of an organism. A most interesting aspect of the metabolism-repair structure is that the replication function (4.8) is already entailed in the original arrow diagram (4.1). On the basis of what are already present in form (4.1), “under stringent but not prohibitively strong conditions, such replication essentially comes along for free.” No infinite iteration is required; the hierarchy of repairs may be folded into a closed loop.

Note that no “inverse” has yet been mentioned up to this point. I have used a generic symbol β for the replication map. There are many ways to construct β from nothing else but what is already in the arrow diagram (4.1). Rosen used the simplest way, an inverse evaluation map. The most important aspect of a replication map is that it needs to produce repair functions Φ , which belong to the hom set $H(B, H(A, B))$. Therefore, the codomain of a replication map β must be $H(B, H(A, B))$. The fact that Rosen’s β turns out to be an inverse evaluation map is entirely incidental.

Here is how one constructs Rosen’s β . An element $b \in B$ defines an evaluation map \hat{b} :

Evaluation $\hat{b} : H(B, H(A, B)) \rightarrow H(A, B) \quad \hat{b} \in H(H(B, H(A, B)), H(A, B))$	(4.12)
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by

$$\hat{b}(\Phi) = \Phi(b) \quad (4.13)$$

The function \hat{b} is invertible if it is monomorphic; viz. for every pair of repair maps, $\Phi_1, \Phi_2 \in H(B, H(A, B))$,

$$\hat{b}(\Phi_1) = \hat{b}(\Phi_2) \Rightarrow \Phi_1 = \Phi_2 \quad (4.14)$$

i.e.,

$$\Phi_1(b) = \Phi_2(b) \Rightarrow \Phi_1 = \Phi_2 \quad (4.15)$$

The implication of (4.15) is a condition on the repair maps $\Phi \in H(B, H(A, B))$. If two repair maps agree at b , then they must agree everywhere. In other words, a repair map Φ (gene) is uniquely determined by its one value $\Phi(b) \in H(A, B)$ (enzyme). This result may be regarded as the abstract representation of the *one-gene-one-enzyme hypothesis*. These are essentially the “stringent but not prohibitively strong conditions” required to make the inverse evaluation map a replication map, with nothing but the ingredients of arrow diagram (4.1).

The inverse evaluation map \hat{b}^{-1} maps thus:

Inverse evaluation $\hat{b}^{-1} : H(A, B) \rightarrow H(B, H(A, B)) \quad \hat{b}^{-1} \in H(H(A, B), H(B, H(A, B)))$

(4.16)

with

$$\hat{b}^{-1}(\Phi(b)) = \Phi \tag{4.17}$$

It takes one image value $\Phi(b) \in H(A, B)$ to the whole function $\Phi \in H(B, H(A, B))$. This is the sense in which it “replicates.” The stringent condition requiring that a repair map Φ to be uniquely determined by the one value $\Phi(b)$ in its range neatly overcomes this $\Phi = \Phi(b)$ identification problem!

For a formal system to model “closure to efficient causation” in a natural system, the simultaneous presence of hierarchy and cycle in its inferential entailment structure is necessary. I will now show that the three functions in an (M,R)-system

$$f \in H(A, B), \tag{4.18}$$

$$\Phi \in H(B, H(A, B)), \quad \text{and} \tag{4.19}$$

$$\beta = \hat{b}^{-1} \in H(H(A, B), H(B, H(A, B))), \tag{4.20}$$

representing the efficient causes of metabolism, repair, and replication respectively, do form a hierarchical closed loop. This is the property that leads to Corollary 3.1.

The hierarchical part is immediately apparent from a simple inspection of the domains and codomains in the three lines (4.18), (4.19), and (4.20). To see the closed loop part, we need to study in greater detail the entailment patterns of mappings.

The map (4.18) takes an element $a \in A$ in its domain, and maps it to the output $b = f(a) \in B$ in its range. The efficient cause and output relationship may be characterized as “ f entails b ” (Secs. 5H and 9D in Ref. 12). I denote this entailment as

$$f \vdash b \tag{4.21}$$

Similarly, the map (4.19) is $\Phi(b) = f$, with the entailment

$$\Phi \vdash f \tag{4.22}$$

The map (4.20) is $\beta(f) = \beta(\Phi(b)) = \hat{b}^{-1}(\Phi(b)) = \Phi$ [see (4.17)]. The fact that $\beta = \hat{b}^{-1}$ means β is uniquely determined by b , and this establishes the correspondence between the sets $H(H(A, B), H(B, H(A, B)))$ and B . The entailment $\beta \vdash \Phi$ may therefore be replaced by the isomorphic

$$b \vdash \Phi \tag{4.23}$$

The cyclic entailment pattern when we combine lines (4.21), (4.22), and (4.23) is the desired closed loop. The three maps entail one another in a cyclic permutation.

5. Goertzel’s Construction

Mathematical logic dictates that it is not necessary to examine an angle trisection scheme in every detail to know that it cannot possibly work. Similarly, it is not necessary to dissect a computable model to know that it cannot possibly be an (M,R)-system. Thus, any claim that an algorithm is exactly that is doomed from the outset. As I mentioned before, sometimes it takes a little effort to untangle a convoluted counterexample to show where it fails. In the remainder of this paper, I shall show where Goertzel’s model goes wrong.

Goertzel [2] does not dispute Rosen’s development of the (M,R)-system. Rather, he presumes that it is valid. All he disputes is that the resulting maps form an incomputable structure, and he attempts to prove his claim by constructing an algorithm that implements the structure. He starts by stating that “*B* is the set of nodes in Webmind,” without defining what set *A* is. He then says, “ $H(A, B)$ is then the set of mappings from nodes into nodes.” I may either assume $A = B$, or that they are two different sets of nodes. It makes no difference in the subsequent discussions. Thus, Goertzel’s version of Rosen’s metabolism function (4.5) is

$$\boxed{\text{Node mapping } f : A \rightarrow B \quad f \in H(A, B)} \tag{5.1}$$

(By another name, node mapping = node transformer.)

However, Goertzel’s construction diverges from an (M,R)-system in his next step: “we can regard a node as an operator on node mappings, by the logic \hat{n} (node mapping)=node mapping(node).” In other words, for a node $n \in A$ and a node mapping $f \in H(A, B)$,

$$\hat{n}(f) = f(n) \tag{5.2}$$

“The node’s action on a mapping is to tell you what the mapping maps the node into”; he calls such a mapping \hat{n} , which takes node mappings in to nodes, a “node transformer projector.” Note the domain and codomain of \hat{n}

$$\boxed{\begin{array}{l} \text{Node transformer projector} \\ \hat{n} : H(A, B) \rightarrow B \quad \hat{n} \in H(H(A, B), B) \end{array}} \tag{5.3}$$

Compare form (5.3) with Rosen’s evaluation map (4.12). The discrepancy in the domains and codomains is a fundamental structural difference between the two developments.

He then continues with defining a “node awakener” as the inverse of a “node transformer projector” (hence taking nodes into node mappings).

$$\boxed{\begin{array}{l} \text{Node awakener} \\ \hat{n}^{-1} : B \rightarrow H(A, B) \quad \hat{n}^{-1} \in H(B, H(A, B)) \end{array}} \tag{5.4}$$

Compare form (5.4) with Rosen’s inverse evaluation map (4.16). Again, observe the discrepancy in the domains and codomains.

An assembly of Goertzel's maps into something that resembles an (M,R)-system would give:

$$A \xrightarrow{f} B \xrightarrow{\hat{n}^{-1}} H(A, B) \quad (5.5)$$

Thus, it is apparent that Goertzel went through this whole inverse evaluation map exercise for the “repair” step, rather than for the “replication” step. Using his terminology, instead of his “node[^](node mapping)=node mapping(node)” construction, what he should have used is “node[^](node awakener)=node awakener(node).” Goertzel’s “inverse evaluation map” is a node awakener, which takes nodes into node mappings. In Goertzel’s parlance, Rosen’s “inverse evaluation map” for (M,R)-systems takes node mappings into node awakeners. Goertzel missed Rosen’s structure by one level in the categorical hierarchy.

In an (M,R)-system, the repair functions $\Phi \in H(B, H(A, B))$ are primary ingredients in arrow diagram (4.1). It is their own closed entailment that makes (M,R)-systems special. In other words, in (M,R)-systems, morphisms with codomain $H(B, H(A, B))$ come for free from form (4.1). In Goertzel’s system (5.5), all he has constructed is something with $H(A, B)$ as the codomain.

It is also notable that Goertzel’s construction contains only two maps, f and \hat{n}^{-1} , while Rosen’s (M,R)-system has a hierarchical cycle of three maps, f , Φ , and \hat{b}^{-1} . Rosen gave a summary in Ref. 11 on why (albeit not explicitly) a two-map hierarchical cycle would not work. The category-theoretic argument is, essentially, for the metabolism and repair maps (4.18) and (4.19) to entail each other, i.e., for $f \vdash b$ and $\Phi \vdash f$ to form a two-cycle, one needs to establish a correspondence between Φ and b . This amounts to the identification of the efficient cause (represented by Φ) with the material cause (represented by b) in the mapping $\Phi(b) = f$, which can only happen vacuously.

6. Division Algebra

Goertzel’s “Webmind” construction misses a level in the categorical hierarchy. By itself, this shows that the algebraic structure of his algorithm differs from the algebraic structure of an (M,R)-system. Furthermore, his construction in terms of division algebras misses yet another level in the hierarchy.

His statement that a “vector multiplication operator” is invertible “under reasonable algebraic conditions only for $k = 1, 2, 4, 8$ ” is a paraphrase of:

Theorem 6.1. *The only finite-dimensional real division algebras occur for dimensions $k = 1, 2, 4$, and 8 , corresponding to real numbers, complex numbers, quaternions, and Cayley numbers, respectively.*

A *division algebra* is usually defined as a ring in which every nonzero element has a multiplicative inverse. However, for the purpose of proving the above theorem, this is in fact the wrong definition. With this definition, it is possible to construct “real division algebras” of other dimensions. One needs to use the slightly

stronger

Definition 6.1. A *division algebra* is a ring in which for any element a and any nonzero element b , there exist a unique x such that $a = b^*x$, and a unique y such that $a = y^*b$, i.e., multiplication by any nonzero element is one-to-one.

It is clear that this definition implies the existence of multiplicative inverse for nonzero elements. For *associative* algebras (i.e., when multiplication is associative), the converse implication is also true; the existence of inverses implies multiplication by any nonzero element is one-to-one. However, associativity is required in the simple proof. (I will not define what a *ring* is. Just think of the set of real numbers with its 0 and 1, with addition and multiplication, and all the nice algebraic properties that go along with them.) Multiplication is not required to be commutative, but if it is, the division algebra is called a *field*. Real numbers and complex numbers are fields. Multiplication for quaternions is not commutative. Multiplication for Cayley numbers (octonions) is not even associative.

What do division algebras have to do with (M,R)-systems? The two respective concepts of “inverse” are not even the same. In division algebras, one worries about *multiplicative* inverse, while in (M,R)-systems, it is *functional* inverse. The trick is to make them coincide.

Let $(B, +, *)$ be a division algebra, and choose an element $b \in B$; the “multiplication by a fixed element” function may be defined as

$$m_b : B \rightarrow B \quad (\text{i.e., } m_b \in H(B, B)) \tag{6.1}$$

by $\forall x \in B$,

$$m_b(x) = b^*x. \tag{6.2}$$

[The hom set $H(B, B)$ is in the category of division algebras here, and has all the properties that go with this.] The inverse function

$$m_b^{-1} : B \rightarrow B \quad (\text{hence also } m_b^{-1} \in H(B, B)) \tag{6.3}$$

exists if the element b itself has a multiplicative inverse b^{-1} . In this case, one can equate m_b^{-1} to “multiplication by the fixed element b^{-1} ” (i.e., identify the functional inverse with the multiplicative inverse, $m_b^{-1} = m_{b^{-1}}$), then $\forall x \in B$,

$$m_b^{-1}(m_b(x)) = m_b^{-1}(b * x) = b^{-1} * (b * x) = (b^{-1} * b) * x = 1 * x = x. \tag{6.4}$$

In the $b^{-1} * (b * x) = (b^{-1} * b) * x$ part of line (6.4), I have used the associativity of multiplication. Thus, if one is only concerned with finite-dimensional real division algebras, this rules out $k = 8$ the Cayley numbers. One is left with $k = 1, 2$, and 4.

The fatal error in this division algebra model is that in the “multiplication by a fixed element” function, one necessarily must have the same domain and codomain, both being the division algebra B ; whence $m_b \in H(B, B)$ and $m_b^{-1} \in H(B, B)$ [as in lines (6.1) and (6.3)].

Multiplicative inverse $m_b^{-1} : B \rightarrow B \quad m_b^{-1} \in H(B, B)$	(6.5)
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Thus, we see that “multiplicative inverse” as a map is actually one step even lower in the categorical hierarchy than Goertzel’s “inverse evaluation map” in the previous section. It cannot even model Goertzel’s node awakener (5.4). This map (6.5) is at the level of metabolism, and therefore definitely not a candidate for replication.

In short, what Goertzel has constructed bears superficial similarities to (M,R)-systems, but the hierarchy of the mappings and thus the entailment structures are completely different. As such, I have verified the truth, as anticipated, of the only remaining logical possibility (3), that the statement reported as $\neg P(c)$ is actually *not*; i.e., the counterexample is wrong.

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