Multidimensional time:
A much delayed chapter in a phenomenological calculus

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December 2004

0. Abstract

In the paper "Dissipation, Lorentz metric, and information: A phenomenological calculus of bilinear forms" [A. H. Louie & I. W. Richardson (1986) *Mathematical Modelling* **7,** 227-240], we described how our four-dimensional spacetime arose from a special bilinear form on our "description space". For n =2, the splitting of the 4=2^2 dimensions into 3+1 of space and time is a natural consequence of the theory. For a general $n \geq 2$, the splitting of the n^2 dimensions is $\frac{1}{2}n(n+1)$ for space and $\frac{1}{2}n(n-1)$ for time. In this paper, I revisit the theory that leads to universes with space of more than three dimensions *and* time of more than one dimensions.

1. Introduction

The phenomenological calculus is, in essence, a mathematical tool for the representations of complex systems. Previous published papers in this epistemological journey in the 1980s are [1–9].

The mathematical object of the phenomenological calculus is the *description space* D, a subspace of the space $T_1^1(H)$ of type-(1,1) tensors over a real Hilbert space H. Members of D are dyadics called *response tensors*. I shall try to review enough concepts to make the present paper (more-or-less) self-contained. The reader is invited to refer to the sequence [1–9] for details.

The seed of this line of exploration was the Richardson & Rosen paper [0], investigating aging from the standpoint of dynamical similitude. It was sown in the Red House, the quarters of the Biomathematics Program at Dalhousie University, with which Robert Rosen was associated from 1975 to 1994. Rosen wrote in the preface of the 1985 book *Theoretical Biology and Complexity* [10] (which, incidentally, comprised three essays by Richardson, Louie, and Rosen, respectively, and was offered as a memento to the late great Red House):

> "...for a perhaps brief but precious time, [the Red House program was] what I regard as one of the most innovative and fruitful programs for research and teaching in theoretical biology in the world. ... it is precisely the scientific strengths of such programs that also make them vulnerable, in constant threat of engulfment by the sands of the vast academic deserts which surround them."

The phenomenological calculus has proven to be extremely versatile in its applicability to various biological, physical, and chemical topics: the list beginning with aging [2], enzyme-substrate recognition [3], (M,R)-systems [4], ... After a flurry of activities in the early 1980s. however, the subject has been all but forgotten. The primary reason, of course, is that both Richardson and I have been engulfed by the aforementioned metaphorical sand. A secondary reason is the mathematical metalanguage of multilinear algebra, which is not part of the standard repertoire, even for pure mathematicians. But just like Rosen uses the mathematical metalanguage of category theory in his complexity studies, we choose this particular metalanguage because the subject demands it: any other metalanguage would have been unnatural.

The first paper [1] in the phenomenological calculus sequence established a metric algebra for the quantification of the aging of a system. Thus from the outset, the concept of *time*, and its companion *age*, both enter in the phenomenological calculus as intrinsic components. Befitting the **TIME** theme of this inaugural issue of BioTheory, I shall concentrate on this one topic in this paper (after a review of the notations and concepts).

2. Description space

Let H be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$. Let H^* be its dual space, i.e. the space of all continuous linear functionals on H, with corresponding dual inner product $\langle \cdot,\cdot \rangle^*$. A type-(1,1) tensor on H is a map from $H^* \times H$ to $\mathbb{R}$ which is bilinear, i.e. linear in each of its two arguments. The linear space of type-(1,1) tensors on H is denoted $T_1^1(H)$.

A *dyad* aF, where $a \in H^*$ and $F \in H$, is a special type-(1,1) tensor, with its actions defined by

$$aF(b,G) = \langle a,b \rangle^* \langle F,G \rangle \tag{1}$$

for $b \in H^*$, $G \in H$. Given dyads aF and bG, their *double inner product* is defined via (1), viz.

$$\langle\langle aF, bG \rangle\rangle = \langle a,b \rangle^* \langle F,G \rangle . \tag{2}$$

A *dyadic* is a finite sum of dyads, whence

$$\mathbf{R} = a^i F_i , \tag{3}$$

where $a^i \in H^*$ and $F_i \in H$. A modified Einstein summation convention is used: repeated *upper and lower* Roman indices denote summation, thus

$$a^i F_i = a^1 F_1 + a^2 F_2 + \ldots + a^m F_m \; ; \tag{4}$$

repeated Greek indices denote a particular term, thus $a^\alpha F_\alpha$ is the α th term only. Clearly $\mathbf{R} \in T_1^1(H)$, and the definition of the double inner product may be extended to dyadics. In [3], we showed that the double inner product is in fact well defined on all of $T_1^1(H)$, and indeed makes $T_1^1(H)$ a Hilbert space.

For fixed covariant vectors $a^1, a^2, \ldots, a^m \in H^*$, the dyadics $\mathbf{R} = a^i F_i$ (with $F_1, F_2, \ldots, F_m$ ranging freely in H) span a linear subspace of $T_1^1(H)$. We call this set

$$D = \left\{ \mathbf{R} = a^i F_i : F_i \in H \right\} \tag{5}$$

the *description space* determined by the set of *constitutive parameters* $\{a^i\}$. Members $\mathbf{R}$ of D are called *response tensors*. Thus response tensors are simply dyadics having a specified form $\mathbf{R} = a^i F_i$ for fixed constitutive parameters $\{a^i\}$, and belonging to a specified subspace D of $T_1^1(H)$.

The metric structure of our phenomenological calculus is formally defined by these three postulates (which first appeared in [2]):

Postulate G1. A complex system is characterized by a set of vectors $\{a^i\}$ that depends on the constitution of the system. A system response is then characterized by the specification of the causal action F_i, and of those constitutive properties a^i of the system which are agents of mediation between action and response. The index i denotes (functional) subsystems.

Postulate G2. The system dynamics are characterized phenomenologically by the response tensor.

Postulate G3. The response tensor is invariant under coordinate transformations in the description space.

The phenomenological connection between causes and effects is provided by the geometric structure of description spaces. We *define* effects J^j to be the duals of the causes F_i, using the fact that $\mathbf{R} = a^i F_i$ is, among other things, a linear mapping $H^* \to H^*$, defined by

$$\mathbf{R}(a^j, \cdot) = (a^i F_i)(a^j, \cdot) = \left\langle a^i, a^j \right\rangle^* F_i = L^{ij} F_i \equiv J^j , \tag{6}$$

with phenomenological coefficients

$$L^{ij} = \left\langle a^i, a^j \right\rangle^* . \tag{7}$$

(Readers familiar with irreversible thermodynamics may have recognized that it and our bilinear phenomenology have a similar metric structure. Indeed the latter was motivated by the former. But our phenomenology is a great deal more general.)

Writing **R** in its dual representation

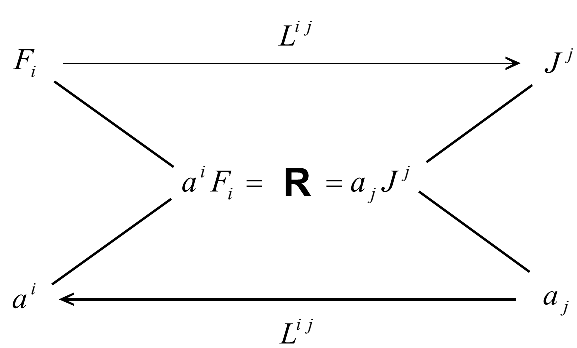
$$\mathbf{R} = a^i F_i = a_j J^j \tag{8}$$

leads to the linear system of equations

$$L^{ij} a_j = a^i \tag{9}$$

for the dual constitutive parameters $a_1, a_2, \ldots, a_m \in H$. The set $\{a_j\}$ would be determined uniquely if the Gram matrix $\left(L^{ij}\right)$ of $\{a^i\}$ were invertible. But this in general is not the case, and we have some degrees of freedom in picking the solution $\{a_j\}$ to (9). This has interesting interpretations in the context of the unidirectionality of causes and effects [read unidirectionality of the arrow of time!]. See [2], in particular, for details.

The metric geometry of our phenomenological calculus may be succinctly expressed in the following arrow diagram, which we call the *duality-invariance diagram* (DID):



(10)

3. Projections

With the phenomenological calculus we have a tool to study the natural world as manifested by phenomena: their genesis, their interrelationships, and in general, the morphology of representations of phenomena. The following four projection postulates (which were first proposed in [5]) provide the link between the hidden and the observable:

Postulate P1. Nature is a Hilbert space over the real field, and it consists of invariants **I**, which are not directly accessible.

Postulate P2. Real manifestations (i.e. observables, phenomena, or appearances) are projections of **I**. (The term *projection* is taken here in its functional analytic sense, that it is an idempotent self-adjoint bounded linear operator. See [5] for details.)

Postulate P3. It is possible to synthesize a representation, **R**, of the relevant features of the invariant, **I**, from measures on the appearances and their projectors. As far as reproducing the appearances, **R** is the best approximation to **I**, in the sense that their metric distance $\|\mathbf{I} - \mathbf{R}\|$ is minimal.

Postulate P4. The representation **R** is the response tensor in our phenomenological calculus, and is invariant in form. In particular, there is a dual representation [which, when shown without the symbolism of projectors, is summarized by form (8) above].

Projection operators (except the trivial ones) are singular. This means that one can never completely reconstruct the original invariant **I** from the response tensors **R**. Another way to state this is that we can never have $\|\mathbf{I} - \mathbf{R}\| = 0$, which is to say $\mathbf{I} = \mathbf{R}$, because a complete description of the invariant is something that we do not have enough information to achieve. This, of course, is an alternate characterization of complexity.

4. Intrinsic time

Our paper [7] has the title "Irreversible thermodynamics, quantum mechanics and intrinsic time scales". We demonstrated there that irreversible thermodynamics and quantum mechanics are homomorphic. In both cases, the metric structure of the phenomenological calculus allowed us to define a proper time τ intrinsic to the dynamics:

$$\left(\frac{d\tau}{dt}\right)^2 = \|\mathbf{R}\|^2 = \langle\langle \mathbf{R}, \mathbf{R} \rangle\rangle = \langle F_i \| J^j \rangle = \delta \geq 0 \,. \tag{11}$$

I won't repeat the details of [7] here. I mention (11) in passing simply to show how time may arise as a characteristic of a metric space.

5. Three special bilinear forms

Our paper [8] studied three symmetric bilinear forms on the description space. They represent dissipation, Lorentz metric, and information. Their mathematical connections equipped our phenomenological calculus as a unified theory of thermodynamics, relativity, and quantum mechanics.

Let **R** and **S** be two response tensors in D. Consider the three bilinear forms

$$\delta(\mathbf{R},\mathbf{S}) = \langle\langle \mathbf{R},\mathbf{S} \rangle\rangle = tr(\mathbf{R}\,\mathbf{S}^T), \tag{12}$$

$$\lambda(\mathbf{R},\mathbf{S}) = \langle\langle \mathbf{R}^T,\mathbf{S} \rangle\rangle = tr(\mathbf{R}\,\mathbf{S}), \tag{13}$$

$$\iota(\mathbf{R},\mathbf{S}) = tr(\mathbf{R})\,tr(\mathbf{S}), \tag{14}$$

where tr denotes the trace function and $\cdot^T$ denotes the transpose function [of the matrix representation of the respective response tensors].

The quadratic forms associated with these three bilinear forms are

$$\delta(\mathbf{R}) = \langle\langle \mathbf{R},\mathbf{R} \rangle\rangle = \| \mathbf{R} \|^2, \tag{15}$$

$$\lambda(\mathbf{R}) = \langle\langle \mathbf{R}^T,\mathbf{R} \rangle\rangle = tr(\mathbf{R}^2), \tag{16}$$

$$\iota(\mathbf{R}) = \left[tr(\mathbf{R}) \right]^2. \tag{17}$$

(We use the same symbols δ, λ, and ι; it is evident from the number of arguments whether the bilinear or the quadratic form is meant.)

It is clear that $\delta \geq 0$ is the dissipation function that also appeared in the definition of intrinsic time in (11). The third form $\iota \geq 0$ turned out to be intimately related to the concepts of recognition and information. I shall leave them aside now, and finally get to the on-topic discussion of multidimensional time, something that is inherent in the form λ.

6. Lorentz metric

Henceforth I shall consider the special Hilbert space $H = \mathbb{R}^n$. This means $H^* = \mathbb{R}^n$, and $T_1^1(H) = \mathbb{R}^{n^2}$. The diagonal form of the matrix representation of the bilinear form λ is

$$\lambda = \operatorname{diag}(+1,...,+1,-1,...,-1), \tag{18}$$

with $\frac{1}{2}n(n+1)$ +1s and $\frac{1}{2}n(n-1)$ -1s. Thus λ has full rank n^2, and signature

$$\tfrac{1}{2}n(n+1) - \tfrac{1}{2}n(n-1) = n . \tag{19}$$

In particular, when $n = 2$,

$$\lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{20}$$

which is the Lorentz metric on Minkowski space, the familiar four-dimensional spacetime metric of relativity.

We thus see that in our four-dimensional spacetime, the 3+1 splitting of space and time is a natural consequence of the theory of a special bilinear form in our phenomenological calculus. Our Lorentz metric (20) is on the four-dimensional "description spacetime" D, constructed from the underlying Euclidean space $\mathbb{R}^2$. See [8] for the remaining relativistic discussions.

7. Multidimensional time

What happens when $n > 2$? What is the significance of $\frac{1}{2}n(n+1)$ space dimensions (+1s) and $\frac{1}{2}n(n-1)$ time dimensions (-1s)? For $n = 3$, there are 6 space dimensions and 3 time dimensions, and for $n = 4$, there are 10 space dimensions and 6 time dimensions. How may we interpret multidimensional time? Let us see what Rosen had to say in the final section (4.8) of the chapter "The Encodings of Time" in his book *Anticipatory Systems* [11]:

> "..., we have abundantly seen that the quality we perceive as time is complex. It admits a multitude of different kinds of encoding, which differ vastly from one another. Thus, we have considered reversible Hamiltonian time; irreversible dynamical time; thermodynamic time; probabilistic time; and sequential or logical time. Each of these capture some particular aspects of our time sense, at least as these

aspects are manifested in particular kinds of situations. While we saw that certain formal relations could be established between these various kinds of time, none of them could be reduced to any of the others; nor does there appear to exist any more comprehensive encoding of time to which all of the kinds we have discussed can be reduced. ..."

So we agree that the standard encoding of "the set of instances" as the continuum $\mathbb{R}$ of real numbers (and for discrete time as its subset $\mathbb{Z}$ of integers) is not sufficient. In our phenomenological calculus, we have the extra degrees of freedom provided by encoding time as the subspace $\mathbb{R}^{\frac{1}{2}n(n-1)}$. In this way, the various kinds of time, which are mutually irreducible, may each be encoded differently into its own continuum $\mathbb{R}$. The experience of time in everyday occurrence is the projection (in the sense of our four projection postulates in Section 3) of this multidimensional time onto $\mathbb{R}$.

There are other theories of multidimensional physics, notably the 10-or-11-dimensional physics of string/brane theory, the latest "theory of everything". But they all assume only *one* time dimension, with the other dimensions occupied by space and hidden variables. In our phenomenological calculus, multidimensional time arises naturally.

8. Songs of the journey

Sir William Rowan Hamilton discovered the quaternions, the first noncommutative division algebra to be studied, in 1843. He felt quaternions would revolutionize mathematical physics — that he had a synthesis of three-dimensional space, the vector-part, and time, the scalar part of the quaternion — and spent the last 22 years of his life studying them. In his poetry he wrote:

> "And how the One of Time, of Space the Three,
> Might in the Chain of Symbols girdled be."

Quaternions do indeed foreshadow the four-dimensional spacetime world of Einstein's relativity. When physicists employ quaternions, however, they often feel that there are not enough dimensions. One extension has been instead of Hamilton's "real quaternions", one replaces the four components of the quadruplet by complex numbers, and ends up with the "biquaternions". But even in this formulation, the time variable is always written as a pure imaginary number ("ict"), so time stays one-dimensional in this alternate description of spacetime.

Arthur Cayley compared the quaternions with a pocket map:

> "...which contained everything but had to be unfolded into another form before it could be understood."

We would like to propose that our phenomenological calculus is one such unfolding. The projection into our four-dimensional spacetime, the dimension of quaternions, may then be interpreted accordingly, in the metalanguage of the multilinear algebra of the description space. It is my sincere hope that this much delayed chapter will rejuvenate our phenomenological calculus.

Hamilton imagined geometry as a science of space, and algebra as a science of time. I will close this chapter of our journey by quoting his 1835 book *Algebra as the Science of Pure Time*:

> "...the subject matter of algebraic science is the abstract notion of time; divested of, or not yet clothed with, any actual knowledge which we may possess of the real Events of History, or any conception which we may frame of Cause and Effect in Nature; but involving, what indeed it *cannot* be divested of, the thought of *possible* Succession, or of pure, *ideal* Progression."

This paper is dedicated to Dr. I. W. Richardson.

9. References

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