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A metabolism–repair theory of by-products and side-effects

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The plurality of process outputs is a genericity of Nature. In this paper, Natural Law receives a new mathematical formulation founded on two axioms: 'Everything is a set.' and 'Every process is a *set-valued* mapping.' I present a brief introduction to the algebraic theory of set-valued mappings, which culminates in two particular morphisms: the metabolism bundle and the imminence mapping. These are relations defined on the collection of processes of a natural system, and serve to characterize material entailment and functional entailment. Generalized metabolism is material entailment of (by-)products, and generalized repair is functional entailment of (side-)effects. Metabolism–Repair networks, hence equipped with set-valued processors, expand their role from models of biological entities to generic models of all natural systems.

Keywords: relational biology; (M,R)-system; by-product; side-effect; set-valued mapping

1. Overture: *The Reflection of Life*

My new monograph *The Reflection of Life: Functional Entailment and Imminence in Relational Biology* (Louie 2013; henceforth denoted by the canonical symbol *RL*) has recently been published by Springer New York. It is a continuation of the exploratory journey in relational biology begun with my previous monograph *More Than Life Itself: A Synthetic Continuation in Relation Biology* (Louie 2009; *ML*). The theme of *ML* is 'What is life?'; the theme of *RL* is 'How do two lifeforms interact?'.

Biology is a subject concerned with organization of relations. Relational biology is the approach that hails 'function dictates structure', rather than 'structure implies function'. It is mathematics decoded into biological realizations.

The cast and crew of mathematical and biological characters in *ML* include partially ordered sets, lattices, simulations, models, Aristotle's four causes, graphs, categories, simple and complex systems, anticipatory systems and metabolism–repair [(M,R)-] systems. In *RL*, the cast and crew are expanded to employ set-valued mappings, adjacency matrices, random graphs and interacting entailment networks.

The theory of set-valued mappings culminates in the imminence mapping, which equips the further investigation of functional entailment in complex relational networks. Imminence in (M,R)-networks that model living systems addresses the topics of biogenesis and natural selection. Interacting (M,R)-networks with mutually entailing processes serve as models in the study of symbiosis and pathophysiology. The formalism also provides a natural framework for a relational theory of virology and oncology.

In this paper, the theory of set-valued mappings is further developed to explicate another aspect of relational pathophysiology (a subject that I first investigated in Louie 2012): the

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general problems connected with system error, malfunction, and breakdown – by-products and side-effects. I shall include herein some background material on set-valued mappings. Otherwise, I assume the reader is already familiar with the premises of the Rashevsky-Rosen school of relational biology. If not, I cordially invite the reader to read both *ML* and *RL* for a comprehensive illustration of the powers of the approach that is relational biology. In particular, as prerequisites for this paper, the reader should know what (M,R)-networks and (M,R)-systems are, be able to distinguish between sequential and hierarchical compositions of mappings and have already understood the following statements.

Definition The entailment of an efficient cause is called *functional entailment*. (*Cf. ML*: Section 5.15)

Definition A natural system is *closed to efficient causation* if its every efficient cause is (functionally) entailed within the system. (*ML*: 6.23)

Postulate of Life A natural system is an *organism* if and only if it realizes an (M,R)-system. (*ML*: 11.28)

Theorem *A natural system is an organism if and only if it is closed to efficient causation.* (*ML*: 11.29)

2. Mapping and its relational diagram

Definition If X is a set, the *power set* $\mathcal{P}X$ of X is the family of all subsets of X.

Definition Given two sets X and Y, one denotes $X \times Y$ the set of all *ordered pairs* of the form (x, y) where $x \in X$ and $y \in Y$. The set $X \times Y$ is called the *product* (or *cartesian product*) of the sets X and Y.

Definition A *relation* is a set R of ordered pairs; i.e. $R \subset X \times Y$ for some sets X and Y.

The category in which the collection of objects is the collection of all sets (in a suitably naive universe of small sets) and where morphisms are relations is denoted **Rel**. Given two sets X and Y, the hom-set $\mathbf{Rel}(X, Y)$ of *all* relations between X and Y is thus the power set $\mathcal{P}(X \times Y)$; whence

$$|\,\mathbf{Rel}(X, Y)\,| = |\,\mathcal{P}(X \times Y)\,| = 2^{|X \times Y|} = 2^{|X|\,|Y|} \tag{1}$$

(where $|\,X\,|$ is the cardinality of the set X).

Definition A *mapping* is a set f of ordered pairs with the property that, if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

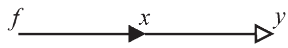
Let X and Y be sets, and let $f \subset X \times Y$ be a mapping. With the conventional requirement that X is the domain of f (notation $X = \mathrm{dom}(f)$), one says that f is a mapping of X into Y (where $Y = \mathrm{cod}(f)$ is the codomain of f), and denotes it by

$$f : X \to Y. \tag{2}$$

The traditional concept of a mapping is that which assigns to *each* element of a given set a definite element of another given set; i.e. a ‘point-to-point’ map. That is, to each input element $x \in X$, by definition there corresponds a unique output element $y \in Y$ such that $(x, y) \in f$. y is called the *value* of the mapping f at the element x, and the relation between x and y is denoted by $y = f(x)$ instead of $(x, y) \in f$. The collection of all the values of f is the range of f (notation $\operatorname{ran}(f)$). To trace the path of an element as it is mapped, one uses the 'maps to' arrow and writes

$$f : x \mapsto y. \tag{3}$$

A mapping thus represented, its relational diagram in graph-theoretic form may be drawn as



(4)

The hollow-headed arrow denotes the *flow* from input $x \in X$ to output $y \in Y$, and the solid-headed arrow denotes the induction of or constraint upon this flow by the *processor* f. The processor (efficient cause) and output (final cause) relationship may be characterized '*f entails y*', which may then be denoted using the entailment symbol $\vdash$ (*RL*: 6.1) as

$$f \vdash y. \tag{5}$$

Note that the processor f is *that which entails*, and the output y is *that which is entailed*.

That which is entailed may take on a secondary role, when f composes with another mapping. In the sequential composite $g \circ f$, the output y of f is used as input (material cause) by another mapping g, whence $\vdash y$ is called *material entailment*. In the hierarchical composite $f \vdash y \vdash$, the output y of f is itself (the efficient cause of) a mapping (functional process), whence $\vdash y$ is called *functional entailment*.

3. The meagreness of single-valued mappings

The category in which the collection of objects is the collection of all sets (in a suitably naive universe of small sets) and morphisms are mappings is denoted **Set**. Given two sets X and Y, one uses the notation Y^X for the hom-set $\mathbf{Set}(X, Y)$ of *all* mappings from X to Y, and

$$|\,\mathbf{Set}(X, Y)\,| = \left|\, Y^X \,\right| = |\,Y\,|^{|X|}. \tag{6}$$

Set is a (non-full) subcategory of **Rel**: the collection of objects are identical for the two categories, and for two sets $X(\neq \varnothing)$ and Y, the hom-sets have a proper containment relationship:

$$\mathbf{Set}(X, Y) \subset \mathbf{Rel}(X, Y) \quad \text{but} \quad \mathbf{Set}(X, Y) \neq \mathbf{Rel}(X, Y). \tag{7}$$

Note the requirement for a subset of $X \times Y$ to qualify as a mapping is in fact quite a stringent one: an element $x \in X$ cannot be related to more than one element of $y \in Y$. Most relations, i.e. *generic* members of $\mathcal{P}(X \times Y)$, do not have this 'single-valued' property. From (1) and (6), one sees that the proportion of mappings to relations is

$$\frac{|\,\mathbf{Set}(X, Y)\,|}{|\,\mathbf{Rel}(X, Y)\,|} = \frac{|\,Y\,|^{|X|}}{2^{|X|\,|Y|}} = \left(\frac{|\,Y\,|}{2^{\,|Y|}}\right)^{|X|}. \tag{8}$$

This is an exceedingly small number; for example, even for trivially small sets X and Y with $|\,X\,| = |\,Y\,| = 10$, there are only $|\,\mathbf{Set}(X, Y)\,| = 10^{10}$ mappings among

$|\,\mathbf{Rel}(X,Y)\,| = 2^{100} \approx 10^{30}$ relations, so the proportion of mappings to relations is about 10^{-20}. Indeed, in the limit, one has

$$\frac{|\,\mathbf{Set}(X,Y)\,|}{|\,\mathbf{Rel}(X,Y)\,|} \to 0 \quad \text{as} \quad |\,X\,| \to \infty \text{ or } |\,Y\,| \to \infty. \tag{9}$$

4. Set-valued mapping

Definition A A *set-valued mapping* from set X to set Y is a relation $F \subset X \times Y$.

Rel, the category of sets and relations, is thus also the category in which the collection of objects is the collection of all sets (in a suitably naive universe of small sets) and morphisms are set-valued mappings.

A set-valued mapping may be denoted

$$F : X \multimap Y \tag{10}$$

such that for each $x \in X$,

$$F(x) = \{y \in Y : (x,y) \in F\} \subset Y, \tag{11}$$

Note the *point-to-set* nature of a set-valued mapping, as opposed to 'point-to-point' for a standard mapping. The 'value' $F(x)$ may contain more than one element, and it is possible that for some $x \in X$, one has $F(x) = \varnothing$.

Since a set-valued mapping $F : X \multimap Y$ takes its values in the family of subsets of Y (i.e. the power set $\mathcal{P}Y$ of Y), one may alternatively consider

Definition B A *set-valued mapping* from set X to set Y is a (single-valued) mapping $F : X \to \mathcal{P}Y$.

In terms of Definition A, the number of all set-valued mappings from set X to set Y is $|\,\mathbf{Rel}(X,Y)\,| = |\,\mathcal{P}(X \times Y)\,| = 2^{|X \times Y|} = 2^{|X|\,|Y|}$. In terms of Definition B, the number is $|\,\mathbf{Set}(X,\mathcal{P}Y)\,| = \left|\,(\mathcal{P}Y)^X\,\right| = |\,\mathcal{P}Y\,|^{\,|X|} = \left(2^{\,|Y|}\right)^{|X|}$. The two enumerations are, naturally, identical.

I have invented in *RL* the special 'forked arrow' $\multimap$ to denote set-valued mappings, in distinction from $\to$ for a standard (single-valued) mapping. Although I often (but do not necessarily) use capital letters to denote set-valued mappings (e.g. F in $F : X \multimap Y$) and lower-case to denote standard mappings (e.g. f in $f : X \to Y$), the two species of arrows are the determinants that distinguish the formal causes. The same symbolic representations suffice for the other arrow diagrams; context determines the nature of the final cause, whether it is an 'element', a 'set' or some other entity. Thus, for $x \in X$ and $E = F(x) \subset Y$, in the set-valued mapping's element-tracing form, one may write

$$F : x \mapsto E. \tag{12}$$

The processor and output relationship may likewise be characterized '*F entails E*', which may then be denoted using the entailment symbol $\vdash$ as

$$F \vdash E. \tag{13}$$

The input of F is, as for a standard mapping, still a point $x \in X$, but now the output of the mapping F at the element x is a *set* $E = F(x) \subset Y$. The source (material cause) and the value (final cause) of a set-valued mapping are thus different in kind from each other, they belonging to different hierarchical levels ('point' versus 'set'). The property of 'that which is entailed' is inherited by elements from their containing set: if F entails E, F also entails every member of E. This is the logical statement

$$F \vdash E \Rightarrow \forall\, y \in E \quad F \vdash y. \tag{14}$$

The *domain* and *codomain* of the set-valued mapping $F : X \multimap Y$ are, respectively, the sets X and Y, denoted by $\mathrm{dom}(F) = X$ and $\mathrm{cod}(F) = Y$. The *range* of F is

$$\mathrm{ran}(F) = \{y \in Y \ : \ \exists\, x \in X \ \ y \in F(x)\}, \tag{15}$$

which may also be expressed as

$$\mathrm{ran}(F) \ = \bigcup_{x \in X} F(x) \subset Y. \tag{16}$$

A standard (single-valued) mapping $f : X \to Y$ may be considered a very specialized set-valued mapping $F : X \multimap Y$ such that, for each $x \in X$, the value

$$F(x) = \{f(x)\} \tag{17}$$

is a singleton set. Indeed, one can make the formal definition: a set-valued mapping $F : X \multimap Y$ is called *single-valued* if, for each $x \in X$, $F(x)$ is a singleton set. A 'single-valued set-valued mapping' $F : X \multimap Y$ therefore defines a 'standard' mapping $f : X \to Y$ by $f : x \mapsto$ the single element in $F(x)$.

5. Image, inverse and sequential composites

Definition Let F be a set-valued mapping from X to Y. If $E \subset X$, the *image* of E under F is defined as the set

$$F(E) = \bigcup_{x \in E} F(x) \subset Y. \tag{18}$$

Definition Given a set-valued mapping $F : X \multimap Y$, its *inverse* is the set-valued mapping $F^{-1} : Y \multimap X$ (equivalently, the relation $F^{-1} \subset Y \times X$) defined by interchanging the ordered components in the graph of F ($= \{(x, y) \in X \times Y : y \in F(x)\}$):

$$F^{-1} = \{(y,x) \in Y \times X : y \in F(x)\} = \{(y,x) \in Y \times X : (x,y) \in F\}. \tag{19}$$

A (single-valued) mapping $f : X \to Y$ is not necessarily injective, so its inverse is not necessarily single valued and hence not (well defined as) a mapping. And even when f^{-1} does exist, its domain is not necessarily the whole codomain Y of f, but only $\mathrm{ran}(f)$ (although f^{-1} is both injective and surjective thence). The inverse $F^{-1} : Y \mathrel{-\!\!\sqsubset} X$ of a set-valued mapping $F : X \mathrel{-\!\!\sqsubset} Y$, contrariwise, is always a set-valued mapping, the two having the exact inverses for domains and codomains. Note, however, that F^{-1} is itself a point-to-set mapping (not a 'set-to-point mapping', as a direct reversal-of-roles 'inverse' of a point-to-set mapping would have been), with its value at the point $y \in Y$ defined as the set

$$F^{-1}(y) = \{x \in X : (x,y) \in F\} \subset X. \tag{20}$$

Indeed, since both $F(x)$ and $F^{-1}(y)$ are defined by the membership $(x,y) \in F$, one trivially has the equivalence $y \in F(x)$ if and only if $x \in F^{-1}(y)$.

Let $F : X \mathrel{-\!\!\sqsubset} Y$ and $G : Y \mathrel{-\!\!\sqsubset} Z$ be set-valued mappings. Their sequential composite may be defined in two different ways:

(i) The *sequential composition* is the set-valued mapping $G \circ F : X \mathrel{-\!\!\sqsubset} Z$ defined by, for $x \in X$,

$$(G \circ F)(x) = \bigcup_{y \in F(x)} G(y) \subset Z. \tag{21}$$

(ii) The *square product* is the set-valued mapping $G \square F : X \mathrel{-\!\!\sqsubset} Z$ defined by, for $x \in X$,

$$(G \square F)(x) = \bigcap_{y \in F(x)} G(y) \subset Z. \tag{22}$$

Note that *the codomain of F is the domain of G*, enabling the two compositions that are the sequence '*F* followed by *G*'. Also note the symbols used for the two binary operations: for sequential composition in (21) it is the standard 'small circle' $\circ$ of 'composite'; for square product in (22), it is a 'small square' $\square$.

While the inverse operator on set-valued mappings is an involution,

$$\left(F^{-1}\right)^{-1} = F, \tag{23}$$

it is important to note that $F^{-1} : Y \mathrel{-\!\!\sqsubset} X$ is not the inverse of $F : X \mathrel{-\!\!\sqsubset} Y$ in the operational sense: neither $F^{-1} \circ F$ nor $F^{-1} \square F$ is necessarily the identity map $1_X : X \mathrel{-\!\!\sqsubset} X$. This is in marked contrast to the inverse mapping f^{-1} (when it exists) of a (single-valued) mapping $f : X \to Y$, for which $f^{-1} \circ f = 1_X$ (and $f \circ f^{-1} = 1_{\mathrm{ran}(f)}$). The not-an-operational-inverse property of F^{-1} will turn out to have significant consequences when it comes to the potential reversal of the entailment action of F.

As relations, subsets of $X \times Z$, the two sequential composites (i) and (ii) are

$$(G \circ F) = \{ \ (x,z) \in X \times Z : \ \exists \, y \in Y \ \ (x,y) \in F \ \wedge \ (y,z) \in G \ \} \tag{24}$$

and

$$(G \,\square\, F) = \{(x,z) \in X \times Z : \ \forall \, y \in Y \ \ (x,y) \in F \Rightarrow (y,z) \in G \ \}. \tag{25}$$

From the $\exists - \text{versus} - \forall$ characterization, one may consider that, for each element $x \in X$ as it is mapped by F into Y and then by G into Z, the sequential composition $(G \circ F)(x)$ traces *at least one* path while the square product $(G \,\square\, F)(x)$ traces *all* such paths. In either case, when one has $z \in (G \circ F)(x)$ or $z \in (G \,\square\, F)(x)$ (equivalently, when one has $(x,z) \in (G \circ F)$ or $(x,z) \in (G \,\square\, F)$), the implication is that for at least one element $y \in F(x)$, the element-trace relay $x \mapsto y \mapsto z$ is possible.

Since $A \cap B \subset A \cup B$ and '$\forall \prec \exists$', one may be tempted to conclude from the definitions of the two sequential composites that $(G \,\square\, F)(x) \subset (G \circ F)(x)$. But the situation is more subtle, and it depends on whether $|F(x)| > 1$, $= 1$, or $= 0$.

The containment $(G \,\square\, F)(x) \subset (G \circ F)(x)$ is indeed true when $|F(x)| \geqslant 1$. When $|F(x)| > 1$, the containment is often proper. It may incidentally happen that for $a, b \in F(x)$, $G(a) \cup G(b) \neq \varnothing$ whence $(G \circ F)(x) \neq \varnothing$, but $G(a) \cap G(b) = \varnothing$ whence $(G \,\square\, F)(x) = \varnothing$. In such a case, there are paths to relay $x \mapsto F(x) \mapsto (G \circ F)(x)$, but there is no common destination for all paths beginning with x and processed sequentially by F then G. One therefore sees that the square product $(G \,\square\, F)(x)$ is more stringent in its relay than the sequential composite $(G \circ F)(x)$.

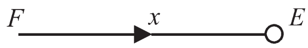
When $|F(x)| = 1$, the union in (21) and intersection in (22) are both taken over the same single element, hence $(G \,\square\, F)(x) = (G \circ F)(x)$. For standard (single-valued) mappings, for each $x \in X$ $|F(x)| = 1$, the two sequential composites (i) and (ii) therefore coincide (and are identical to the standard sequential composition of mappings).

When $|F(x)| = 0$, viz. $F(x) = \varnothing$, one has to be aware of 'empty set pathologies' from taking the union and intersection over an empty set. In a lattice L, the least element and greatest element (when they exist) are $\inf L = \sup \varnothing$ and $\sup L = \inf \varnothing$. For the lattice $\mathcal{P}Z$, one has $\varnothing = \inf \mathcal{P}Z = \bigcup\limits_{\varnothing}$ and $Z = \sup \mathcal{P}Z = \bigcap\limits_{\varnothing}$ (*cf. ML*: 1.28). Therefore, in the singular scenario of $F : x \mapsto \varnothing$, when F acting on input x produces no outputs, $(G \circ F)(x) = \bigcup\limits_{y \in \varnothing} G(y) = \varnothing$ and $(G \,\square\, F)(x) = \bigcap\limits_{y \in \varnothing} G(y) = Z$, which *a fortiori* yields $(G \,\square\, F)(x) \not\subset (G \circ F)(x)$. One may explain the apparent paradox thus. The existence of $z \in (G \circ F)(x)$ implies the existence of an element $y \in F(x)$, for which $z \in G(y)$, to complete the relay path $x \mapsto y \mapsto z$. But $F(x) = \varnothing$ means no such relay point $y \in F(x)$ can exist, so contrapositively there can be no $z \in (G \circ F)(x)$; whence $(G \circ F)(x) = \bigcup\limits_{y \in \varnothing} G(y) = \varnothing$. On the other hand, for an element $z \in Z$ to satisfy $z \in (G \,\square\, F)(x)$, it must happen that whenever $y \in F(x)$, the relay $x \mapsto y \mapsto z$ ensues. But $F(x) = \varnothing$ means the implication '$y \in F(x) \Rightarrow$ relay $x \mapsto y \mapsto z$' is vacuously true (i.e. there is no $y \in F(x)$ to contradict the statement), so every $z \in Z$ qualifies; whence $(G \,\square\, F)(x) = \bigcap\limits_{y \in \varnothing} G(y) = Z$.

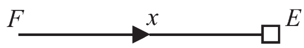
Part I of *RL* is a pentateuchal exploration of the algebraic theory of set-valued mappings. It also contains the motivations and other natural philosophical reasons on why I would consider them congenial and congenital morphisms for relational biology. The enthused reader is invited to consult *RL* for further details on this much-neglected topic in mathematics.

6. Relational diagrams

In Chapter 3 of *RL*, with the cognate reasoning of context-driven distinction as in the forms (12) and (13), I have used the same symbology (4) of ⟨solid-headed arrow + hollow-headed arrow⟩ pair for the relational diagram of a set-valued mapping. This latter formal cause, however, did not make much of a reappearance after its debut. I now, in anticipation of the exploration to come, think it is appropriate to introduce variations on the theme. Instead of the 'hollow-*triangle*-headed arrow' of a single-valued mapping, I propose new formal causes in both a 'hollow-*circle*-headed arrow' and a 'hollow-*square*-headed arrow' for 'that which is entailed' in the set-valued mapping $F : x \mapsto E$ (where $x \in X$ and $E = F(x) \subset Y$):



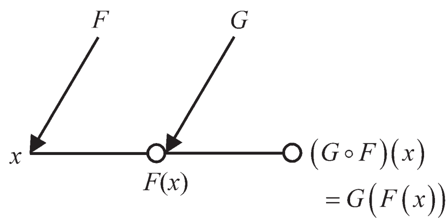
(26)



(27)

Both the circle- and square-headed species indicate that the final cause (output) is a set, and the two represent the two different kinds of compositions that may be involved.

The relational diagram of sequential composition $(G \circ F)(x)$ is



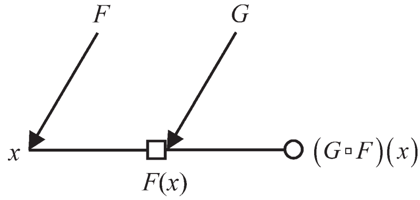
(28)

The crucial node in the entailment network (28) is the 'relay vertex' $F(x)$, where the two set-valued mappings $F : X \multimap Y$ and $G : Y \multimap Z$ interact and their relational diagrams connect. That $F(x)$ is where its hollow-circle-headed arrow and the solid-headed arrow of G meet represents the execution of the *union* operation

$$\bigcup_{y \in F(x)} G(y), \tag{29}$$

which is the very definition of $(G \circ F)(x)$. The iconography of this connection, stated otherwise, symbolizes a sequential relay of *at least one* output in $F(x)$ as a material cause of *G*; at least one elemental path must pass through here to reach $(G \circ F)(x)$. The hollow-circle-headed arrow that terminates on $(G \circ F)(x) = G(F(x))$ simply indicates that the final output is a set, but it could have been replaced by a hollow-square-headed arrow. It is the further relay of this output that would determine the alternatives of circle and square.

The relational diagram of square product $(G \square F)(x)$ is



(30)

Here, the node $F(x)$ is where its hollow-square-headed arrow and the solid-headed arrow of G meet; the connection represents the execution of the *intersection* operation.

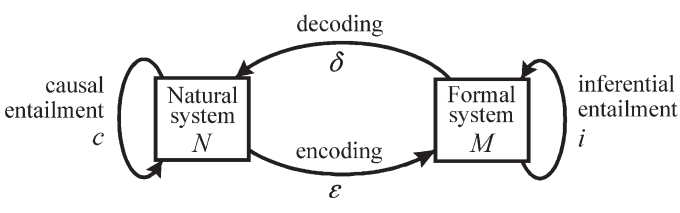
$$\bigcap_{y \in F(x)} G(y), \tag{31}$$

which is the very definition of $(G \square F)(x)$. This iconography, then, symbolizes a sequential relay of *all* the outputs in $F(x)$ as a material causes of G; all elemental traces through the relay vertex $F(x)$ must terminate at the final output $(G \square F)(x)$. As before, the hollow-circle-headed arrow that terminates on the final output simply indicates that the latter is a set, with a possibility of symbol change if this output is further relayed.

7. Mathematical formulation of natural law

Causality is the principle that every effect has a cause, and is a reflection of the belief that successions of events in the world are governed by definite relations. *Natural Law* posits the existence of these *entailment* relations *and* that this causal order can be *imaged* by implicative order.

A *modelling relation* is a commutative functorial encoding and decoding between two systems. Between a natural system (an object partitioned from the physical universe) N and a formal system (an object in the universe of mathematics) M, the situation may be represented in the following canonical diagram:



(32)

The encoding ε maps the natural system N and its causal entailment c therein to the formal system M and its internal inferential entailment i; i.e.

$$\varepsilon : N \mapsto M \text{ and } \varepsilon : c \mapsto i. \tag{33}$$

The decoding δ does the reverse. The entailments satisfy the commutativity condition that, in diagram (32), tracing through arrow c is the same as tracing through the three arrows ε, i, and δ in succession. This may be symbolically represented by the 'composition'

$$c = \delta \circ i \circ \varepsilon. \tag{34}$$

Thence related, M is a *model* of N, and N is a *realization* of M. In terms of the modelling relation, then, Natural Law is a statement on the existence of causal entailment c and the encodings $\varepsilon : N \mapsto M$ and $\varepsilon : c \mapsto i$.

A formal system may simply be considered as a *set* with additional mathematical structures. So the mathematical statement $\varepsilon : N \mapsto M$, i.e. the posited existence for every natural system N a model formal system M, may be stated as the axiom

$$\textit{Everything is a set}. \tag{35}$$

Causal entailment in a natural system is a network of interacting processes, i.e. a network of efficient causes. The mathematical statement $\varepsilon : c \mapsto i$, i.e. the functorial correspondence of morphisms, between causality c in the natural domain and inference i in the formal domain, may thus be stated as an epistemological principle, the axiom

$$\textit{Every process is a mapping}. \tag{36}$$

Together, the two axioms (35) and (36) are the mathematical foundation of Natural Law. These self-evident truths serve to explain 'the unreasonable effectiveness of mathematics in the natural sciences'.

The collection of all models of a natural system N is denoted $\mathbf{C}(N)$ (*ML*: 7.27). $\mathbf{C}(N)$ is a lattice (*ML*: 7.28) as well as a category (*ML*: 7.29). Let $\kappa(N)$ be the collection of all efficient causes in N. An entailment network that models N may be denoted $\varepsilon(N) \in \mathbf{C}(N)$, and the collection of all efficient causes in the network $\varepsilon(N)$ may be represented by the encoding $\varepsilon(\kappa(N))$. Natural Law is the statement.

$$\forall N \;\; \exists\, \varepsilon \;\; \exists M \in \mathbf{C}(N) \;\; : \;\; M = \varepsilon(N) \;\; \wedge \;\; \forall c \in \kappa(N) \;\; \exists\, i \in \varepsilon(\kappa(N)) \;\; : \;\; i = \varepsilon(c). \tag{37}$$

For notational simplicity, however, one often drops the encoding symbol ε and uses N to denote both the natural system and its network model that is a formal system. Thus 'an entailment network $\varepsilon(N)$ that models a natural system N' abbreviates to 'an entailment network N'. Likewise, the symbol $\kappa(N)$ shall denote the collection of efficient causes in both the natural system and the formal system. These identifications $\varepsilon(N) = N$ and $\varepsilon(\kappa(N)) = \kappa(N)$ amount to an implicit invocation of Natural Law. The relational diagram of $\kappa(N)$ is a digraph representing the entailment network N (*RL*: Chapter 9).

8. Every process is a set-valued mapping

Axiom (36) carries with it the baggage inherited from Newtonian physics, which mathematically is the analysis of single-valued mappings. In view of the meagreness (8) and (9) of single-valued mappings among set-valued mappings, I now rephrase axiom (36) as

$$\textit{Every process is a set-valued mapping}. \tag{38}$$

Stated otherwise, I am expanding our theatre from the category **Set** to the category **Rel**. Because of the containment $\mathbf{Set}(X, Y) \subset \mathbf{Rel}(X, Y)$, however, a set-valued mapping may incidentally be single-valued, so the new extension encompasses the old foundation. The category $\mathbf{C}(N)$ of all models of a natural system N is now a subcategory of **Rel**.

The extension of the Natural Law axiom from (36) to (38) realizes the fact that processes inevitably entail more than their single, primary, 'intended' outputs. When the extraneous secondary outputs are material causes, they are called *by-products*; when they are efficient causes, they are called *side-effects*. Although these two terms often have negative connotations, and are predominantly employed to describe adverse outputs, they can as well apply to beneficial, albeit still unanticipated, consequences. In the next two sections, I shall illustrate with a few examples.

It is important to note my theory of 'set-valuedness' of process outputs has nothing to do with fuzziness or probabilistic reasoning. «Je n'avais pas besoin de cette hypothèse-là.» There is no difference in determinacy between set-valued mappings and their specialized single-valued comrades, neither necessitating stochastics in their formulations. Axiom (38), 'Every process is a set-valued mapping.', has to do with the well connectedness of the entailment networks of Nature, that every action entails a plenitude of consequences. (Think 'butterfly effect'.)

> **Intention,** *n*. The mind's sense of the prevalence of one set of influences over another set; an effect whose cause is the imminence, immediate or remote, of the performance of an involuntary act.
>
> – Ambrose Bierce (1911) *The Devil's Dictionary*

The teleology of 'One. True. Purpose.' of a natural process is an artefact. A process simply is; its plural outputs simply are. Members of an output set (i.e. constituents of a 'value' of a set-valued mapping) are coextensive. All those materially entailed are products; all those functionally entailed are effects. The existence of a meta-process of value judgement is implicit when one distinguishes a 'desired' product and adds the prefix 'by-' to the rest, and likewise when one isolates an 'intended' effect and relegates others to be qualified with 'side-'. Given a set-valued mapping $F : X \multimap Y$, the teleological assignment of a single final cause is a *choice mapping*

$$\varepsilon \ : \ \{F(x) \subset Y : x \in X\} \to Y \tag{39}$$

that selects, for each $x \in X$, from the set $F(x) \subset Y$ a single value

$$\varepsilon(F(x)) \in F(x) \subset Y; \tag{40}$$

the procedure thus defines a single-valued mapping

$$f = \varepsilon \circ F \ : \ X \to Y. \tag{41}$$

(See *RL*: 0.20 for the definition of choice mapping, and *RL*: 9.3 for its role in an explication of functional closure.) The use of the same symbol ε for the encoding functor and the choice mapping is not coincidental: the single-purpose specialization

$$\hat{\varepsilon} \;:\; F \;\mapsto\; f \tag{42}$$

is the encoding

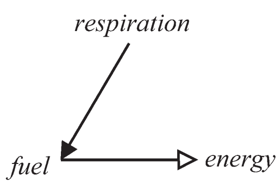
$$\hat{\varepsilon} \;:\; \kappa(N) \to \hat{\varepsilon}(\kappa(N)) \tag{43}$$

in the modelling of a natural system N with set-valued processes by its abstraction with single-valued processes.

As with any model, the commitment to a specific choice mapping ε, whence a particular encoding $\hat{\varepsilon}$ projecting set-valued mappings F to single-valued mappings f, loses information in its execution. In this case, the loss of 'degrees of freedom', 'closing' each set $F(x)$ in the restriction to a single output $\varepsilon(F(x)) \in F(x)$, renders unentailed all the by-products and side-effects, which consequently become unexplainable in the decoding. To wit, the encoding is a simple choice (a possible invocation of the Axiom of Choice notwithstanding; *cf. RL*: 0.20 & 1.2) of a single element $y_x = \varepsilon(F(x)) \in F(x)$ from each member of a family $\{F(x) : x \in X\}$ of subsets of Y; but there is no trivial procedure for the from-one-to-all extrapolation $\delta : \{y_x : x \in X\} \to \mathcal{P}Y$. The decoding, from a paucity of ingredients, has the futile task to reconstruct from each member y_x of the indexed set $\{y_x : x \in X\}$ to a superset $\delta(y_x)$, such that $y_x \in \delta(y_x) \subset Y$ and the reconstitution $F(x) = \delta(y_x)$ recovers the original set-valued mapping $F : X \multimap Y$. This loss of entailment closes functionally open systems, and this informational incompleteness is, indeed, what Robert Rosen proposed as the cause of side-effects (Rosen 1985). I shall discuss this further in Section 10 below.

9. By-products

Cellular respiration is the metabolic process that takes place in the cells of organisms to convert biochemical energy from nutrients into a useful form to fuel cellular activities. It may be succinctly represented by this relational diagram:

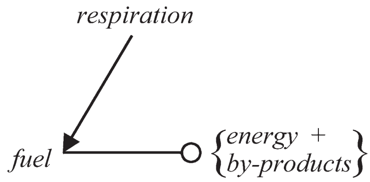


(44)

The reductionistic study of this process (or set of processes) forms the bulk of the curriculum of biochemistry. There is no need here to get into the intricacies of glycolysis, citric acid cycle, Krebs cycle, etc. It suffices to state (aerobic) respiration in its simplest form:

$$C_6H_{12}O_6 + 6\,O_2 \to 6\,CO_2 + 6\,H_2O + \textit{energy} \tag{45}$$

Water is useful (in moderation) in the cellular environment, while carbon dioxide is a toxin, and higher organisms have elaborate respiration systems for its removal. Both H_2O and CO_2 are by-products for the *raison d'être* of respiration that is useful-energy generation. The relational diagram (44) is thus more accurately represented as a set-valued mapping:

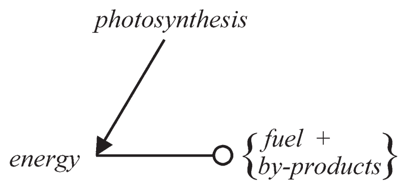


(46)

Photosynthesis is a process used by plants (and other organisms) to convert light energy into chemical energy that can be used to fuel the organisms' activities. Biochemistry students also study, of course, photosynthesis in terms of the biochemical pathways of the Calvin cycle. The simplest chemical equation for oxygenic photosynthesis is

$$2n\, CO_2 + 2n\, H_2O + \textit{light energy} \rightarrow (C\,H_2O)_{2n} + 2n\, O_2 \tag{47}$$

Photosynthetic organisms are photoautotrophs: they are able to synthesize food directly from carbon dioxide and water using light energy. Oxygen is released, mostly as a waste product (of the photosynthetic process itself; oxygen is otherwise crucial in the biosphere for all the oxygen-breathing organisms). In effect, photosynthesis is the 'inverse process' of cellular respiration, and its set-valued mapping has this relational diagram:



(48)

It is a testament to the wonder of Nature that two complimentary processes exist in the biosphere to precisely use each other's by-products as input material causes. One must note, however, that the two processes take place through a different sequence of chemical reactions and in different cellular compartments. While photosynthesis and cellular respiration are mutual teleological inverses in terms of entailment, they do not 'reverse' each other's efficient causes. In short, as set-valued mappings,

$$\textit{photosynthesis} \;\neq\; (\textit{respiration})^{-1}. \tag{49}$$

Other examples of (positive and negative) by-products include: manure from animal husbandry, straw from grain harvesting, the plethora of 'petroleum products' from the refining of crude oil to produce gasoline, radioactive waste from nuclear power generation and 'green house gases' from combustion. The reader may easily append to this list.

10. Side-effects

The term 'side-effect' is borrowed from medicine, and describes unavoidable and usually unfortunate consequences of employing therapeutic agents. Robert Rosen has considered the subject in Rosen (1974a and 1974b) from the viewpoint of 'how planning could go wrong', and devoted a substantial portion of the introductory Section 1.1 of *Anticipatory Systems* (Rosen 1985) on a repeat discussion. In the latter, a control-theoretic definition of 'side-effect' is found:

> unplanned and unforeseeable consequences on system behavior arising from the implementation of controls designed to accomplish other purposes; or, in a related context, the appearance of unpredicted behavior in a system built in accordance with a particular plan or blueprint

followed by Rosen's explication:

> ... the ultimate seat of the side-effects arising in anticipatory control, and indeed of the entire concept of error or malfunction in system theory as a whole, rests on the discrepancy between the behavior actually exhibited by a natural system, and the corresponding behavior predicted on the basis of a model of that system. For a model is necessarily an *abstraction*, in that degrees of freedom which are present in the system are absent in the model. In physical terms, the system is *open* to interactions through these degrees of freedom, while the model is necessarily *closed* to such interactions; the discrepancy between system behavior and model behavior is thus a manifestation of the difference between a closed system and an open one.

Thus, Rosen contends that side-effects are necessarily entailed from the exercise of control. Note the adjectives 'unplanned', 'unforeseeable' and 'unpredicted' in Rosen's definition are all *model-dependent* qualifiers. The categorization of certain outputs as '*un*anticipated consequences', as *side-effects* (and likewise as *by*-products), is an artificial value-judgement, a subjective imposition of 'purpose', an extrinsic choice in model-making (*cf.* my explication in Section 8 above).

Rosen's epistemological explanation of the exodus of side-effects, as perused from Rosen (1974a) and (1974b), is that it is a natural consequence of every model abstraction, entailed from 'something relevant has been missed'. (And, depending on the quality of the chosen predictive models, some models miss more than others.) I am in this paper offering a complementary, ontological explanation, that the genesis of side-effects is a natural consequence of the multitudinous outputs of every process, entailed from 'something extraneous is concurring'. Both explanations are, nevertheless, ultimately anchored on the same premise that natural processes multitask, that they entail polysemously, that, irrevocably, 'Every process is a set-valued mapping.'

The ideal of the *magic bullet*, a substance or therapy capable of providing a remedy for an ailment without deleterious side-effects, is not, in general, achievable. It is also unlikely that side-effects can be removed by simply augmenting the model, or by attempting to control each side-effect separately as it appears. (Think of the stereotypical cartoon of the Little Dutch Boy who stuck a finger into a leaking hole in a dike to stem the flow of water, only to have another leak sprung up elsewhere, and eventually ran out of additional digits to plug an ever-increasing number of holes.) This is because generic natural systems are complex (in the Rosen sense; *cf. ML*: Chapter 9). Only simple models have simulable largest models, which, when used, produce no side-effects, because thence every entailed effect has been accounted for. Indeed, all aspects of simple system behaviour, from generation to revelation, are explicitly embodied in the finitely generated syntax of that largest model. At times, it is possible to avoid the divergence of an infinite sequence of patchwork fixes in complex system modelling; this is when certain functional closure in the system can be implemented. Closure to efficient causation and its corollaries are not, however, the subject of this paper; these have been addressed in *ML* and *RL*.

Saint Bernard of Clairvaux wrote (*c*. 1150) «L'enfer est plein de bonnes volontés et désirs.» Side-effects are the curse of chemical therapeutics. There is no need to dwell herein on this well-known failure, except to note that almost all therapeutic and diagnostic agents have side-effects. A spectacular example, to give but one for illustration, of such paved road to hell is the thalidomide disaster in the late 1950s and early 1960s. Occasionally some side-effects are, however, beneficial. For instance, acetylsalicylic acid (ASA, 'Aspirin'), which is a pain reliever primarily (which really only signifies a historical accident that its analgesic properties were discovered first), has a secondary unintended effect as an anticoagulant that can help prevent heart attacks and reduce the severity and damage from thrombotic strokes. A beneficial side-effect, once discovered, may lead to the drug's functional change, so that the secondary effect becomes one of the primary ones. ASA is now widely prescribed in its preventive role.

The introduction of a non-native species into an ecosystem for an intended purpose (pest control, decoration, recreation, ...) often 'upsets the balance of nature' (when the introduced species flourishes due to the lack of natural predators in the new environment), and does more harm than good. Think of rabbits in Australia, and Africanized bees in the Americas. Many evolutionary changes in organisms are unintended consequences, side-effects of otherwise-purposed processes. This is the 'Principle of Function Change', and Rosen's favourite example (Rosen 1974a) is how the swim bladders of fishes (intended as an organ of equilibration, a 'flotation device') evolved into lungs for air breathers.

Many social and economic policies introduced with good intentions have not only generated unfortunate side-effects, but have actually in the long run served to exacerbate the very problems they were meant to control. Examples of such perverse functional entailments abound: prohibition entailed alcohol-smuggling organized crime, highways without curves induced road hypnosis, an attempt to censor or remove a certain piece of information (document, photograph, video, etc.) from the internet instead caused the information to 'go viral' and became widely known and distributed. The reader is invited to supply more examples of positive and negative side-effects.

11. The imminence mapping

Let N be a natural system. In the category **Rel**, without restriction, models of N are built on **Rel**-objects (sets) and **Rel**-morphisms (set-valued mappings). In general, our models are drawn from smaller non-full subcategories **C** of **Rel**, in which **C**-objects are a selection of sets $A, B, \ldots$, and **C**-hom-sets are proper subsets of $\mathbf{Rel}(A, B)$:

$$\mathbf{C}(A, B) \subset \mathbf{Rel}(A, B) = \mathcal{P}(A \times B). \tag{50}$$

Inclusion (50) is the category-theoretic formulation of Axiom (38), that 'Every process is a set-valued mapping.'

Within this chosen categorical universe of models, one has the following

Definition A *model of* N *in the category* **C** is a member of $\mathbf{C}(N)$ with **C**-objects and **C**-morphisms.

Let $\mathcal{O}\mathbf{C}$ be the collection of **C**-objects (that are sets) and $\mathcal{A}\mathbf{C}$ be its collection of **C**-morphisms (that are set-valued mappings). A model of N in the category **C** may alternatively be described as a formal system that is a network of mappings in $\mathcal{A}\mathbf{C}$, in which case one may alternately refer to 'a system N in the category **C**' when its collection $\kappa(N)$ of efficient causes is a subset of **C**-morphisms:

$$\kappa(N) \subset \mathcal{A}\mathbf{C}. \tag{51}$$

Definition The *imminence mapping of the category* **C** (also the *imminence mapping on* **C**) is the set-valued mapping

$$\mathrm{Imm}_{\mathbf{C}} : \mathcal{A}\mathbf{C} \mathrel{-\!\!\sqsubset} \mathcal{A}\mathbf{C} \tag{52}$$

defined, for $f \in \mathcal{A}\mathbf{C}$, by

$$\mathrm{Imm}_{\mathbf{C}}(f) = \mathcal{A}\mathbf{C} \cap \mathrm{ran}(f). \tag{53}$$

A nonempty set $\mathrm{Imm}_{\mathbf{C}}(f)$, being the collection of all **C**-morphisms that lie in the range of f, is the collection of all the f-entailed entities that can themselves entail, i.e. all possible further actions arising from f, whence the *imminence* of f. This is a key concept, indeed *the* key concept, in *RL*.

Instead of the whole collection $\mathcal{A}\mathbf{C}$ of **C**-morphisms, consider a system N that is a network of mappings in $\mathcal{A}\mathbf{C}$ (e.g. an (M,R)-network), whence the collection $\kappa(N)$ of all efficient causes in N is a subset of $\mathcal{A}\mathbf{C}$, viz. $\kappa(N) \subset \mathcal{A}\mathbf{C}$. The *imminence mapping of the system N in the category* **C** (also the *imminence mapping on* $\kappa(N)$) is the set-valued mapping

$$\mathrm{Imm}_{N} : \kappa(N) \mathrel{-\!\!\sqsubset} \kappa(N) \tag{54}$$

defined, for $f \in \kappa(N)$, by

$$\mathrm{Imm}_N(f) = \kappa(N) \cap \mathrm{ran}(f). \tag{55}$$

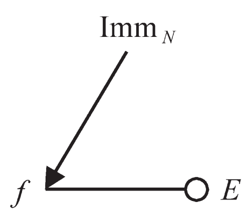
The set $\mathrm{Imm}_N(f)$ is the collection of all efficient causes of N that lie in the range of f, i.e. all the f-entailed entities in $\kappa(N)$. The imminence mapping Imm_N on $\kappa(N)$ *is* the functional entailment pattern of the model of the natural system N. (When $\kappa(N) = \mathcal{A}\mathbf{C}$, e.g. when N is the whole category **C**, one has $\mathrm{Imm}_N = \mathrm{Imm}_{\mathbf{C}}$.)

Let $f \in \kappa(N)$ and $E = \mathrm{Imm}_N(f) \subset \kappa(N)$. Trivially, the corresponding arrow diagrams are

$$\mathrm{Imm}_N \;:\; f \mapsto E \tag{56}$$

$$\mathrm{Imm}_N \vdash E \tag{57}$$

and



(58)

Let $g \in E = \mathrm{Imm}_N(f)$. It is evident from definition (55) of the set-valued mapping $\mathrm{Imm}_{N} : \kappa(N) \mathrel{-\!\!\sqsubset} \kappa(N)$ (and definition (16) of range) that

$$\begin{aligned} g \in \mathrm{Imm}_N(f) \;\Rightarrow\;& g \in \mathrm{ran}(f) \\ \Leftrightarrow\;& \exists\, x \in \mathrm{dom}(f) : g \in f(x) \subset \mathrm{ran}(f). \end{aligned} \tag{59}$$

This means

$$f \;:\; x \mapsto f(x), \tag{60}$$

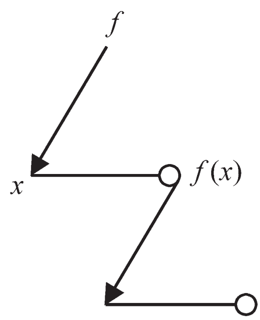
$$f \vdash f(x), \tag{61}$$

and, since $g \in f(x)$, hereditarily according to equation (14) above one has the functional entailment

$$f \vdash g. \tag{62}$$

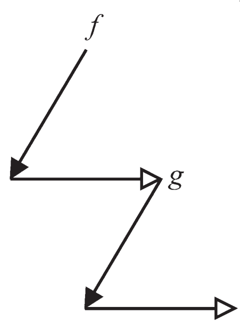
This is to say, for $f, g \in \kappa(N)$, $g \in \mathrm{Imm}_N(f)$ if and only if f functionally entails g.

Set-valued hierarchical entailment (60) has the relational diagram



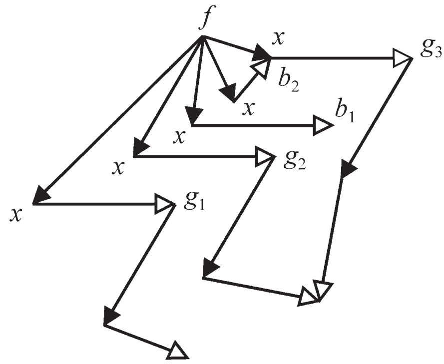
(63)

Contrast this with relational diagram (28) of the sequential composite $(G \circ F)(x)$, in which the relay vertex $F(x)$ is where the *heads* of a hollow-circle-headed arrow and a solid-headed arrow meet. Here in (63), it is the head of a hollow-circle-headed arrow and the *tail* of a solid-headed arrow that meet at the relay vertex $f(x)$. The hollow-circle heads herein just serve to denote the set-valuedness of the mappings involved, and not to distinguish sequential composition from the square product, as in relational diagrams (28) versus (30). Since $f(x)$ is a set, this connection iconography is in fact an 'abbreviation' that represents an ensemble of potentially divergent processes. All that is required is that there is *at least one* $g \in f(x)$, for which the hierarchical composition $f \vdash g \vdash$ proceeds, whence diagram (63) implies the canonical relational diagram of a hierarchical composition:



(64)

As to other members of the set $f(x)$, they may be functionally entailed, with each having its own associated ⟨solid-headed arrow + hollow-headed arrow⟩ pair, or they may be simple material outputs. For example, for a particular $f(x) = \{b_1, b_2, g_1, g_2, g_3\}$, the local element-trace network may look like this:



(65)

It is important to contrast mapping $f : x \mapsto f(x)$ in (60) with mapping $\mathrm{Imm}_N : f \mapsto E$ in (56). They are alternate descriptions ultimately of the functional entailment $f \vdash g$ in (62), but the two senses of 'membership' $g \in f(x)$ and $g \in E = \mathrm{Imm}_N(f)$ are hierarchically different. The mapping $f : x \mapsto f(x)$ is f *itself*, whence (tautologically) with domain = dom(f) and f as its *efficient* cause, while the imminence mapping $\mathrm{Imm}_N : f \mapsto E$ is *about* the entailment among processes in $\kappa(N)$, with $\mathrm{dom}(\mathrm{Imm}_N) = \kappa(N)$ and f as its *material cause*. Also, $f \vdash f(x)$ is set-valued functional entailment (in the summary interpretation of relational diagram (63)). On the other hand, $\mathrm{Imm}_N \vdash E$ may be considered material entailment of the output set E (which just so happens to be a collection of efficient causes); indeed, in the next section, we shall see how imminence Imm_N is iterated in sequential composites (while entailing iterated hierarchical composites of constituent processes).

Rosen arranged the relational organization in his (M,R)-systems in such a way that *repair* is a mapping that produces as output metabolism that is itself also a mapping: for the 'R' part, instead of just producing an entity on which to operate, it could produce an operator, the 'M' part. The essence of an (M,R)-network N is the 'repair $\vdash$ metabolism' functional entailment, which may now alternatively be phrased in terms of the imminence mapping as 'Imm_N: repair $\mapsto$ metabolism'.

The power of the imminence mapping formulation is immediately apparent when one sees that the 'What is life?' answer, that a natural system is an organism if and only if it is closed to efficient causation (and if and only if it realizes an (M,R)-system), may be succinctly characterized in terms of the imminence mapping in the

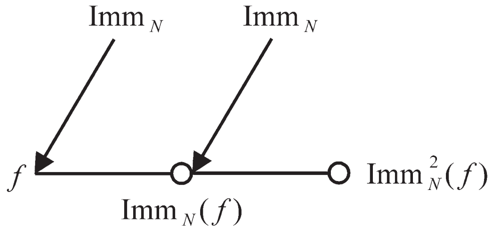
Theorem (*RL*: 9.2) *A system N is closed to efficient causation if and only if, for every* $f \in \kappa(N)$, $\mathrm{Imm}_N^{-1}(f) \neq \varnothing$.

12. Iterated imminence and hierarchical composition

For $f \in \kappa(N)$, the sequential composite $\mathrm{Imm}_N^2 = \mathrm{Imm}_N \circ \mathrm{Imm}_N : \kappa(N) \multimap \kappa(N)$ is defined by

$$\mathrm{Imm}_N^2(f) = (\mathrm{Imm}_N \circ \mathrm{Imm}_N)(f) = \bigcup_{g \in \mathrm{Imm}_N(f)} \mathrm{Imm}_N(g) \subset \kappa(N). \tag{66}$$

This has the relational diagram



(67)

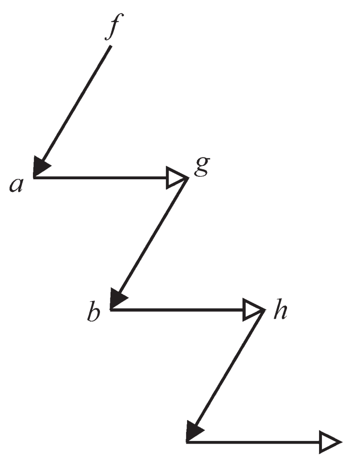
(*cf.* diagram (28) above). The iterated imminence $h \in \mathrm{Imm}_N^2(f)$ entails the existence of an intermediary mapping $g \in \mathrm{Imm}_N(f)$ in the imminence of f such that

$$f \vdash g \vdash h. \tag{68}$$

Symbolically, this situation may be summarized

$$h \in \mathrm{Imm}_N^2(f) \vdash (\exists g \in \mathrm{Imm}_N(f) \ : f \vdash g \vdash h). \tag{69}$$

The three mappings f, g, h form a hierarchical chain



(70)

It is crucial to distinguish between the *sequential composite* $\mathrm{Imm}_N \circ \mathrm{Imm}_N$ of the set-valued imminence mapping Imm_N (relational diagram (67)) and, entailed by the iterated imminence $h \in \mathrm{Imm}_N^2(f) = \mathrm{Imm}_N \circ \mathrm{Imm}_N(f)$, the *hierarchical chain* $f \vdash g \vdash h$ among the mappings f, g and h (relational diagram (70)). In category-theoretic terms, the correspondence $h \in \mathrm{Imm}_N \circ \mathrm{Imm}_N(f) \ \mapsto \ \langle f \vdash \cdot \vdash h \rangle$ is a functor that translates sequential composition into hierarchical composition. In concert with what was discussed in the previous section, here diagrams (67) and (70) are alternate descriptions of the functional entailment chain (68). But in diagram (70) f is an efficient cause, while 'meta-diagram' (67) is *about* the (iterated) imminence of f, with f as material cause.

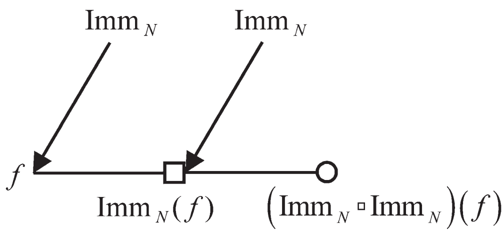
Note that only the existence of one intermediary g in (68) is required, because of the '$\exists$ at least one path' characterization of the sequential composite $\mathrm{Imm}_N \circ \mathrm{Imm}_N$. The mapping f may functionally entail many more mappings in $\kappa(N)$, but none of these other branches are obliged to immediately connect to h. For example, if $g' \in \mathrm{Imm}_N(f)$ and $g' \neq g$, it may well happen that one has $h' \in \mathrm{Imm}_N(g') \subset \mathrm{Imm}_N^2(f)$, $h' \neq h$, but $h \notin \mathrm{Imm}_N(g')$. Stated otherwise, iterated imminence is *divergent*.

The iterated imminence $\mathrm{Imm}_N^2(f)$ may be interpreted as all the processes in the natural system N that are reachable from the process f after two functional entailment steps. This 'indirect imminence' is inherent in the imminence network Imm_N, and has proven to be a relational characterization of viruses (*cf. RL*: Chapter 13 on relational virology).

The square product $\mathrm{Imm}_N \,\square\, \mathrm{Imm}_N : \kappa(N) \multimap \kappa(N)$ is defined by

$$(\mathrm{Imm}_N \,\square\, \mathrm{Imm}_N)(f) = \bigcap_{g \in \mathrm{Imm}_N(f)} \mathrm{Imm}_N(g) \subset \kappa(N). \tag{71}$$

Its relational diagram is

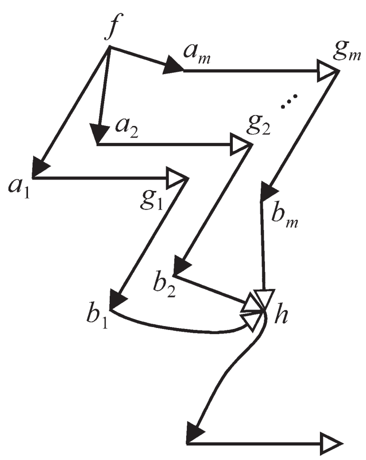


(72)

(*cf.* diagram (30) above). A mapping $h \in (\mathrm{Imm}_N \,\square\, \mathrm{Imm}_N)(f)$, different from the iterated imminence $h \in \mathrm{Imm}_N^2(f)$, must be reachable from the mapping f after travelling on *every* two connected arrow pairs initiating from f in the digraph Imm_N – *all* the mappings $g_1, g_2, \ldots, g_m \in \mathrm{Imm}_N(f)$ must entail h:

$$f \vdash g_1 \vdash h, \quad f \vdash g_2 \vdash h, \ldots, \quad f \vdash g_m \vdash h. \tag{73}$$

This means a relational diagram that contains the branching pattern



(74)

In contrast to immanent causation $\exists g$ in (69), square product imminence may symbolically be summarized as

$$h \in (\mathrm{Imm}_N \,\square\, \mathrm{Imm}_N)(f) \vdash (\forall g \in \mathrm{Imm}_N(f) \ : f \vdash g \vdash h). \tag{75}$$

Stated otherwise, square product imminence is *convergent*. The imposition on all intermediaries $g \in \mathrm{Imm}_N(f)$ entailed by $h \in (\mathrm{Imm}_N \,\square\, \mathrm{Imm}_N)(f)$ is an inherent *redundancy*, a multiplicity of entailment paths that says something about the importance of h to require such protection and robustness to ensure its imminent repair. Repair, in its most general terms, is set-valued functional entailment.

The iterated imminence $h \in \mathrm{Imm}_N^2(f)$ allows a wider branching in the functional entailment network: the existence of one hierarchical composite $f \vdash g \vdash h$, that one requisite output g of f entails h, places no entailment restrictions on any other $g' \in \mathrm{Imm}_N(f)$ (so 'side-effect' may run amok in extent and depth). Contrariwise, in the square product imminence $h \in (\mathrm{Imm}_N \,\square\, \mathrm{Imm}_N)(f)$, all outputs of f (i.e. *all* effects, 'side-effects' included) are more 'reined in', mandated to at least include h in their imminence. Among the plethora of possible network connections, the generic is the 'interior' rather than the 'boundaries'. (This is the 'Goldilocks principle'.) Thus, for iterated imminence, both extremes of single-path relays ($f \vdash g \vdash h$) and all-path relays ($h \in (\mathrm{Imm}_N \,\square\, \mathrm{Imm}_N)(f)$) are exceptions, and the moderation ($h \in \mathrm{Imm}_N^2(f) = (\mathrm{Imm}_N \circ \mathrm{Imm}_N)(f)$) is the rule.

13. The metabolism bundle

Metabolism is all the processes that occur within a living organism, a relay network of metabolites. The *telos* of metabolism is energy production, but with by-products galore. Metabolism, in its most general terms, is set-valued material entailment.

Consider the set-valued mapping

$$\mathrm{Met}_{\mathbf{C}} : \mathcal{A}\mathbf{C} \multimap \mathcal{A}\mathbf{C} \tag{76}$$

defined by

$$\mathrm{Met}_{\mathbf{C}} = \{\ (f,g) \in \mathcal{A}\mathbf{C} \times \mathcal{A}\mathbf{C} \ \ : \ \ \mathrm{dom}(g) \cap \mathrm{ran}(f) \neq \varnothing\ \}. \tag{77}$$

The subset $\mathrm{Met}_{\mathbf{C}} \subset \mathcal{A}\mathbf{C} \times \mathcal{A}\mathbf{C}$ is the domain on which 'metabolism' in $\mathbf{C}$ may proceed, containing pairs of processes (f,g) that may participate in the relay $x \mapsto f(x) \mapsto g(f(x))$. Hence the expression 'Met' as the symbol, and the name *metabolism bundle* that I have given it. (For an explanation of the usage of 'bundle', see *RL*: 10.5.) The analogy between Met and Imm is more apparent if I rephrase the set-valued mapping $\mathrm{Imm}_{\mathbf{C}}$ also as a subset of $\mathcal{A}\mathbf{C} \times \mathcal{A}\mathbf{C}$:

$$\begin{aligned} \mathrm{Imm}_{\mathbf{C}} &= \{\ (f,g) \in \mathcal{A}\mathbf{C} \times \mathcal{A}\mathbf{C} \ \ : \ \ g \in \mathrm{ran}(f)\ \} \\ &= \{\ (f,g) \in \mathcal{A}\mathbf{C} \times \mathcal{A}\mathbf{C} \ \ : \ \ \{g\} \cap \mathrm{ran}(f) \neq \varnothing\ \}. \end{aligned} \tag{78}$$

Similar to the imminence mapping, the metabolism bundle may likewise be restricted to a model of a natural system N in the category $\mathbf{C}$, as a 'model-specific' set-valued mapping

$$\mathrm{Met}_N : \kappa(N) \multimap \kappa(N) \tag{79}$$

defined by

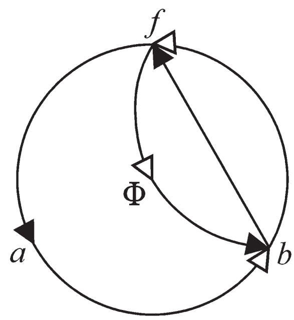
$$\mathrm{Met}_N = \{(f,g) \in \kappa(N) \times \kappa(N) : \mathrm{dom}(g) \cap \mathrm{ran}(f) \neq \emptyset\}, \tag{80}$$

which is the embodiment of the material entailment structure in N.

14. Metabolism–repair system

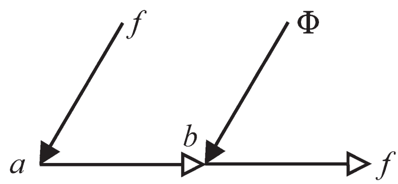
The two subsets Met_N and Imm_N of $\kappa(N) \times \kappa(N)$, i.e. *metabolism* and *repair* in the natural system N, are not necessarily disjoint. The range of a mapping may contain both materially-entailed and functionally-entailed entities. A single output set of a set-valued mapping may itself already contain both species. It may also happen that a single output entity takes on *dual roles* of being materially entailed in one interaction and functionally entailed in another.

As an illustrative example of the latter, consider the simplest (M,R)-system (*cf.* *ML*: Chapters 11 and 12). Its relational diagram is



(81)

The metabolism map $f : a \mapsto b$ and repair map $\Phi : b \mapsto f$ compose sequentially thus:

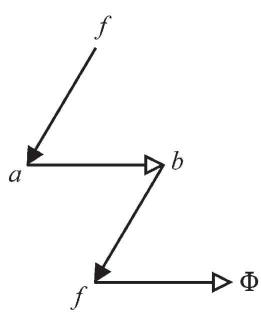


(82)

whence b is materially entailed by f and relayed in

$$\Phi \circ f : a \mapsto b \mapsto f. \tag{83}$$

On the other hand, the metabolism map $f : a \mapsto b$ and the replication map $b : f \mapsto \Phi$ compose hierarchically thus:



(84)

whence b is functionally entailed by f and relayed in

$$f \vdash b \vdash \Phi. \tag{85}$$

So one sees that the output entity b takes on dual roles: that of a materially entailed product in (82) and (83), and of a functionally entailed effect in (84) and (85).

Final causes of processes are not ends in themselves. In the entailment network $\kappa(N)$ of a natural system N, processes are in composition, and among them final causes are further relayed as material and efficient causes. The entailment network $\kappa(N)$ is completely described by two set-valued mappings defined on it: the metabolism bundle Met_N generates (by-)products through material entailment, and the imminence mapping Imm_N generates (side-)effects though functional entailment. Every natural process in $\kappa(N)$ may be categorized as either 'metabolism' or 'repair', even when N is not necessarily a metabolism–repair network per se. Together, Met_N and Imm_N may be taken as the very definition of the entailment network that models the natural system N. As we have just seen in the (M,R)-system example, the two set-valued mappings are not necessarily mutually exclusive. After all, by-products and side-effects are, but, different names that denote the same multifarious outputs, those which are entailed, of interacting processes.

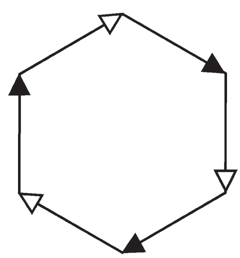
15. Metabolic entailment between systems

The efficient cause of entailment can just as commonly arise without as within. Consider two systems H and S in a category $\mathbf{C}$, whence there are two sets of efficient causes (i.e. two collections of set-valued mappings), $\kappa(H)$ and $\kappa(S)$, that are subsets of $\mathcal{A}\mathbf{C}$. Let g and f be processes in H and S respectively; i.e. $g \in \kappa(H)$ and $f \in \kappa(S)$. If $x \in \text{dom}(f)$ is such that $\text{dom}(g) \cap f(x) \neq \varnothing$, then $\exists\, y \in f(x)$ and the relay

$$x \mapsto y \mapsto g(y) \tag{86}$$

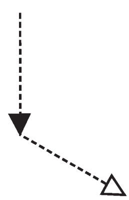
can proceed.

I shall use the canonical hierarchical cycle



(87)

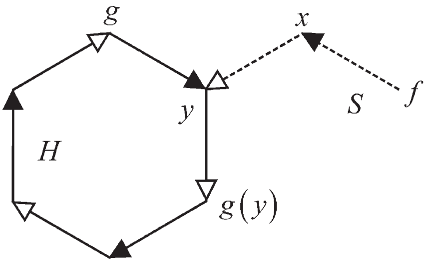
to symbolically represent the system H, and a single arrow pair



(88)

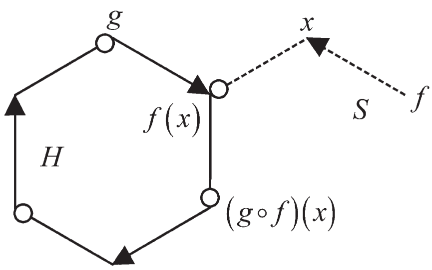
to denote the system S. Of course, the entailment networks of both H and S are far more complicated relational diagrams consisting of large numbers of interconnected arrows. But the interactions between diagrams (87) and (88) are sufficient to illustrate the modes of interactions that I shall discuss. Also, I shall be varying the hollow 'arrowheads' (triangle for element-trace, circle for sequential composition, and square for square product) to represent the different entailment patterns.

The elemental relay (86) is the $H-S$ interaction in the *join* entailment network $N = H \vee S$ (*cf. ML*: 2.1 & 7.28 and *RL*: 11.12 & 13.2):



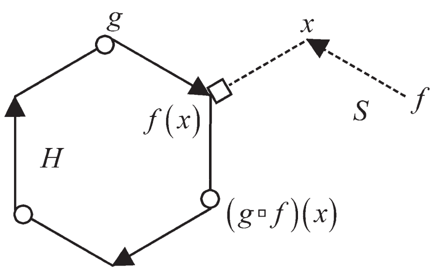
(89)

In Chapter 11 of *RL*, this mode of interaction between H and S is explicated alternatively as symbiosis (when the metabolite y is shared between the two systems) and infection (when the antigen $y \in f(x)$ produced by S invades the host H). The elemental relay (86) is also an instance of sequential composition of the two mappings g and f. Depending on the entailment paths involved, it can either be the sequential composite $g \circ f$:



(90)

or the square product $g \,\square\, f$:



(91)

These two relational diagrams (90) and (91) then serve as models of metabolic interactions between H and S when by-products are involved. Suppose in the original system H, before its interaction with S, the entailment of the mapping $g \in \kappa(H)$ is

$$g \vdash z \tag{92}$$

(with $z \neq g(y)$). The metabolic entailments of the mapping $g \in \kappa(H \vee S)$ in relational diagrams (89), (90) and (91) are, respectively,

$$g \vdash g(y), \quad g \vdash (g \circ f)(x), \quad \text{and} \quad g \vdash (g \,\square\, f)(x). \tag{93}$$

The number of entailment paths involved in the sequential composition of the two mappings g and f increases in these three entailments in (93), from one to at least one to all. This difference in degree may then be realized as a measure of the ease of implementing therapeutic procedures to revert the affected $g \in \kappa(H \vee S)$ back to its native state $g \in \kappa(H)$ with entailment (92).

How the two mappings $g \in \kappa(H)$ and $f \in \kappa(S)$ sequentially compose is completely determined by the join entailment network $\kappa(N) = \kappa(H \vee S)$. Instead of the metabolism bundle $\mathrm{Met}_N \subset \kappa(N) \times \kappa(N)$, one may consider its restriction to the two interacting subsystems H and S. The *metabolism bundle of the action of S on H*, $\mathrm{Met}_{S \to H} : \kappa(S) \multimap \kappa(H)$, takes the form

$$\mathrm{Met}_{S \to H} = \{(f, g) \in \kappa(S) \times \kappa(H) : \mathrm{dom}(g) \cap \mathrm{ran}(f) \neq \varnothing\}. \tag{94}$$

The entailment network $\mathrm{Met}_{S \to H} \subset \kappa(S) \times \kappa(H)$ contains all the metabolic consequences of S on H. Note that $\kappa(S) \times \kappa(H) \subset \kappa(N) \times \kappa(N) \subset \boldsymbol{\mathcal{A}}\mathbf{C} \times \boldsymbol{\mathcal{A}}\mathbf{C}$, and the three metabolism bundles on different domains are related thus:

$$\mathrm{Met}_{S \to H} = \mathrm{Met}_N|_{\kappa(S) \times \kappa(H)} \text{ and } \mathrm{Met}_N = \mathrm{Met}_{\mathbf{C}}|_{\kappa(N) \times \kappa(N)}. \tag{95}$$

16. Functional entailment between systems

The imminence mapping $\mathrm{Imm}_N \subset \kappa(N) \times \kappa(N)$ of the system $N = H \vee S$ may likewise be restricted to the two interacting subsystems H and S. Define a set-valued mapping $\mathrm{Imm}_{S \to H} : \kappa(S) \multimap \kappa(H)$ by, for a mapping $f \in \kappa(S)$,

$$\mathrm{Imm}_{S \to H}(f) = \kappa(H) \cap \mathrm{ran}(f). \tag{96}$$

Hierarchical composition $f \vdash g$ may be defined for $f \in \kappa(S)$ and $g \in \kappa(H)$ if and only if

$$g \in \kappa(H) \cap \mathrm{ran}(f) = \mathrm{Imm}_{S \to H}(f) \neq \varnothing. \tag{97}$$

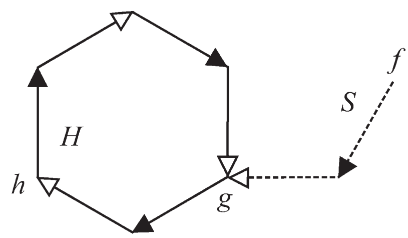
Stated otherwise, $\mathrm{Imm}_{S \to H}(f)$ contains all the processes in the system H that may be functionally entailed by the process $f \in \kappa(S)$ of the system S. The set-valued mapping $\mathrm{Imm}_{S \to H}$ may, therefore, be considered the *imminence of S on H*, i.e. *inter-network imminence*. Functional entailment is *repair* in its most general sense, whence $\mathrm{Imm}_{S \to H}(f)$ may be considered the *repair effect* of the interaction $f : S \to H$. Similar to the three metabolism bundles, the three imminence mappings on different domains are related thus:

$$\mathrm{Imm}_{S\to H} = \mathrm{Imm}_N|_{\kappa(S)\times\kappa(H)} \text{ and } \mathrm{Imm}_N = \mathrm{Imm}_{\mathbf{C}}|_{\kappa(N)\times\kappa(N)}. \tag{98}$$

For $f \in \kappa(S)$ and $g \in \kappa(H)$, the functional entailment $f \vdash g$, i.e.

$$g \ \in \ \mathrm{Imm}_{S\to H}(f), \tag{99}$$

has the relational diagram



(100)

This interaction, the hierarchical composite

$$f \vdash g \quad \text{in} \quad S, \quad g \vdash h \quad \text{in} \quad H \Rightarrow f \vdash g \vdash h \quad \text{in} \quad N = H \vee S, \tag{101}$$

models competing processes between H and S, as well as repair-level and replication-level infections of H by S (*RL*: Chapters 11 and 13). The process $h \in \kappa(H) \subset \kappa(H \vee S) = \kappa(N)$ is reachable from the process $f \in \kappa(N)$ after two functional entailment steps. Thus, depending on the entailment paths involved, it can be formulated in terms of an iterated imminence as either the sequential composite

$$h \in \mathrm{Imm}_N^2(f) \tag{102}$$

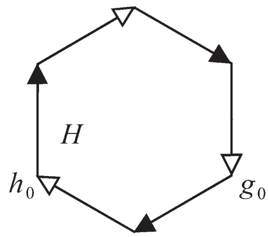
or the square product

$$h \in (\mathrm{Imm}_N \,\square\, \mathrm{Imm}_N)(f) \tag{103}$$

(*cf.* Section 12 above). These two relational diagrams (102) and (103) then serve as models of functional interactions between H and S when side-effects are involved. The original functional entailment

$$g_0 \vdash h_0. \tag{104}$$

in the system H



$$(105)$$

is perturbed by the interaction with S. With increasing number of entailment paths in the hierarchical composition of $f \vdash g \vdash$, the functional entailments involved in (101), (102), and (103) are, respectively,

$$f \vdash g \vdash h, \tag{106}$$

$$f \vdash \mathrm{Imm}_N(f) \vdash h \in \mathrm{Imm}_N^2(f), \tag{107}$$

and

$$f \vdash \mathrm{Imm}_N(f) \vdash h \in (\mathrm{Imm}_N \,\square\, \mathrm{Imm}_N)(f). \tag{108}$$

17. The art of undoing

The plurality of process outputs is a genericity of Nature. All actions have consequences, most of them being side-effects. Therapeutics (medical or otherwise) is the art of undoing actions.

For an illustrative example, consider an attempt in the restoration to the native state (108) of the system H from the interactive state (100) of the join system $N = H \vee S$. A treatment scheme that involves nothing else but the reversal from the perturbed $g \vdash h$ in (101) back to the native $g_0 \vdash h_0$ in (104) is relatively straightforward. But, unfortunately, the singular elemental-tracing relay $f \vdash g \vdash h$ in (101) is a simplified entailment model of the generic hierarchical compositions of set-valued processes that are either (102) or (103). Reductionistic biochemical therapeutics operates on the level of

$$f \vdash g \vdash h \to g_0 \vdash h_0. \tag{109}$$

Relational therapeutics suggests

$$h \in \mathrm{Imm}_{S \to H}^2(f) \to h_0 \in \mathrm{Imm}_H(g_0), \tag{110}$$

on a different hierarchical level.

In the path-tracing model (109), the specification of one intermediary process $g \in \mathrm{Imm}_N(f)$, and all the other processes in $\mathrm{Imm}_N(f)$ have been missed. With only singular $g \in \mathrm{Imm}_N(f)$ and $h \in \mathrm{Imm}_N^2(f)$ (or possibly $h \in (\mathrm{Imm}_N \,\square\, \mathrm{Imm}_N)(f)$) on hand, too much information is lost for the recovery of $g_0 \vdash h_0$. Often, one does not have information on the directly affected g, nor, indeed, on the iterated imminence h. The actual 'observables' ('symptoms') may be some indirect effects, several iterations of hierarchical compositions further along the causal chain.

The relational therapy (110) hinges on knowledge of both the imminence mappings of the healthy state Imm_H and the pathophysiology $\mathrm{Imm}_{S \to H}$, or equivalently, the entailment networks $\kappa(H)$ and $\kappa(H \vee S)$, in which every process is a set-valued mapping. Success depends on the size of the available fragment of imminence. Note the disentanglement suggested in (110) is not the panacea of an 'inverse operation' to 'neutralize' the effect of $f \in \kappa(S)$ on H. Once imminence has been released, laments Lady Macbeth, 'What's done cannot be undone.' Even in the unlikely scenario of complete information (that is to say, when all effects and side-effects due to the imminence of S on H are accounted for), the joining of H and S in $N = H \vee S$ may still not be 'inverted'. Inversion means to have a set-valued mapping $\Theta : \kappa(N) \multimap \kappa(N)$ for which

$$\Theta \circ \mathrm{Imm}_N = 1_{\kappa(N)}. \tag{111}$$

But, as I have explained above in Section 5, (111) is an equation that the only candidate $\Theta = \mathrm{Imm}_N^{-1}$ would most likely *not* satisfy. Some natural processes are inherently irreversible, and attempts at control can only be from prevention, and attempts at remedy are often circuitous (such as the imminence-driven arguments in this paper leading to the functorial (110)).

The imminence mapping $f \mapsto \mathrm{Imm}_N(f)$ encompasses *anticipation*, since the complex relational network $\kappa(N)$ of processes and their entailed potentialities contains internal predictive models of the natural system N and its environment, and in accordance with these predictions antecedent actions are taken and other consequent manifestations follow. My journey in relational biology will continue with the next upcoming episode: *The imminence mapping anticipates.*

Notes on contributor

A. H. Louie is a mathematical biologist. His research subjects have encompassed abstract formulations, mathematical modelling and computer simulations of various natural and physical phenomena, including dynamic behaviour of protein molecules, enzyme–substrate recognition, processes of irreversible thermodynamics, human–pollutant interactions, the cell biology of senescence, and electromagnetics. His premier interest, however, remains the epistemological aspects of mathematical biology. He is glad to have returned to it in 2005 after a 20-year interlude of doing mathematical modelling as a scientist-for-hire. His approach to the subject is called 'relational biology', in the school of Nicolas Rashevsky and Robert Rosen. He has thus far written two books on the subject: *More Than Life Itself: A Synthetic Continuation in Relation Biology* [ontos verlag 2009] and *The Reflection of Life: Functional Entailment and Imminence in Relational Biology* [Springer 2013].

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